Read: Strang, Section 2.4, 2.5, 2.6, 2.7

Suggested short conceptual exercises: Strang, Section 2.4, #2, 7, 11, 14, 32, 33. Section 2.5, #11, 15, 29. Section 2.7, #3–5, 8, 11, 12–16, 19.

1. Let $A$, $B$, and $C$ be $n \times n$ matrices.
   (a) If $A$ is invertible and $AB = AC$, prove that $B = C$.
   (b) Find an example of three nonzero matrices such that $AB = AC$ but $B \neq C$.
   (c) Suppose that $B = C^{-1}AC$ is invertible. Find formulas for $A$, $A^{-1}$, and $B^{-1}$.

2. Let $A$ be a $3 \times 3$ matrix where row 1 + row 2 = row 3. Give elementary answers to the following questions (that is, do not use terms such as “column space” or “nullspace”.)
   (a) Explain why $Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ cannot have a solution.
   (b) Which vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ might allow a solution to $Ax = b$? [Hint: Find a linear equation relating $b_1$, $b_2$, and $b_3$.]
   (c) What happens to row 3 in elimination?

3. Let $A$ be a $3 \times 3$ matrix where column 1 + column 2 = column 3.
   (a) Find a nonzero solution to $Ax = 0$.
   (b) Elimination keeps column 1 + column 2 = column 3. Explain why there is no third pivot. Conclude that $A$ is not invertible.

4. Let $A$ be an invertible $3 \times 3$ matrix, and let $B$ be the matrix obtained from $A$ by taking the bottom row and making it the top row instead (and so the first and second rows of $A$ become the second and third rows of $B$, respectively).
   (a) Find the permutation matrix $P$ that you need to multiply $A$ by to get $B$. Do you need to multiply on the left or on the right?
   (b) What is $P^{-1}$?
   (c) Find simple formulas for $B^{-1}$ and $B^T$.

5. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$.
   (a) What three elementary matrices $E_{21}$, $E_{31}$, and $E_{32}$ put $A$ into its upper triangular form, $E_{32}E_{31}E_{21}A = U$?
(b) Multiply by $E_{32}^{-1}, E_{31}^{-1}$ and $E_{21}^{-1}$ to factor $A$ into $A = LU$, where $L = E_{31}^{-1}E_{32}^{-1}E_{21}^{-1}$ is a lower triangular matrix whose entries are the multipliers of elimination and $U$ is an upper triangular matrix with the pivots on the diagonal.

(c) Factor $U = DU_1$, where $D$ is a diagonal matrix that contains the pivots.

(d) Factor $A = LDU_1$, where $L$ is a lower triangular matrix with 1s on the diagonal and whose other entries are the multipliers, $D$ is a diagonal matrix that contains the pivots, and $U_1$ is an upper triangular matrix with 1s on the diagonal.

6. Let $A$ be a $3 \times 3$ matrix. Suppose you want to do the following operations on $A$: Subtract row 1 from row 2, subtract row 1 from row 3, and subtract row 2 from row 3.

(a) First, suppose you want to perform these operations on $A$ simultaneously. Write down a matrix $B$ that you need to multiply $A$ by to achieve this. Does $AB$ or $BA$ yield the desired result? What is $B^{-1}$? [Hint: You should be able to find $B^{-1}$ by inspection, but it’s not quite a simple as flipping the signs of the entries below the diagonal.]

(b) Now, suppose you want to perform these operations on $A$ sequentially. Write down a matrix $E$ that you need to multiply $A$ by to achieve this. It should be the product of three elementary matrices [Hint: It should not the same matrix as $B$ from Part (a)]. What is $E^{-1}$.

(c) Suppose you factor $A = LU$, where $U$ is the upper triangular matrix with the pivots on the diagonal, and $L$ is lower-triangular. What is the relationship between $L$ and $E$?

7. A matrix is symmetric if $A^T = A$. Factor the following matrices into $A = LDL^T$.

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}, \quad \quad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Note that for any symmetric matrix, the $A = LDU$ factorization simplifies to $A = LDL^T$. 