Read: Strang, Section 2.4, 2.5, 2.6, 2.7

Suggested short conceptual exercises: Strang, Section 2.4, #2, 7, 11, 14, 32, 33. Section 2.5, #11, 15, 29. Section 2.7, #3–5, 8, 11, 12–16, 19.

1. Let $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ be $n \times n$ matrices.
   (a) If $\boldsymbol{A}$ is invertible and $\boldsymbol{AB} = \boldsymbol{AC}$, prove that $\boldsymbol{B} = \boldsymbol{C}$.
   (b) Find an example of three nonzero matrices such that $\boldsymbol{AB} = \boldsymbol{AC}$ but $\boldsymbol{B} \neq \boldsymbol{C}$.
   (c) Suppose that $\boldsymbol{B} = \boldsymbol{C}^{-1}\boldsymbol{AC}$ is invertible. Find formulas for $\boldsymbol{A}$, $\boldsymbol{A}^{-1}$, and $\boldsymbol{B}^{-1}$.

2. Let $\boldsymbol{A}$ be a 3×3 matrix where row $1 +$ row $2 =$ row 3. Give elementary answers to the following questions (that is, do not use terms such as "column space" or "nullspace".)
   (a) Explain why $\boldsymbol{Ax} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ cannot have a solution.
   (b) Which vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ might allow a solution to $\boldsymbol{Ax} = \boldsymbol{b}$? [*Hint*: Find a linear equation relating b_1, b_2, and b_3.]
   (c) What happens to row 3 in elimination?

3. Let $\boldsymbol{A}$ be a 3×3 matrix where column $1 +$ column $2 =$ column 3.
   (a) Find a nonzero solution to $\boldsymbol{Ax} = \boldsymbol{0}$.
   (b) Elimination keeps column $1 +$ column $2 =$ column 3. Explain why there is no third pivot. Conclude that $\boldsymbol{A}$ is not invertible.

4. Let $\boldsymbol{A}$ be an invertible 3×3 matrix, and let $\boldsymbol{B}$ be the matrix obtained from $\boldsymbol{A}$ by taking the bottom row and making it the top row instead (and so the first and second rows of $\boldsymbol{A}$ become the second and third rows of $\boldsymbol{B}$, respectively).
   (a) Find the permutation matrix $\boldsymbol{P}$ that you need to multiply $\boldsymbol{A}$ by to get $\boldsymbol{B}$. Do you need to multiply on the left or on the right?
   (b) What is $\boldsymbol{P}^{-1}$?
   (c) Find simple formulas for $\boldsymbol{B}^{-1}$ and $\boldsymbol{B}^T$.

5. Consider the matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$.
   (a) What three elementary matrices $\boldsymbol{E}_{21}$, $\boldsymbol{E}_{31}$, and $\boldsymbol{E}_{32}$ put $\boldsymbol{A}$ into its upper triangular form, $\boldsymbol{E}_{32}\boldsymbol{E}_{31}\boldsymbol{E}_{21}\boldsymbol{A} = \boldsymbol{U}$?

(b) Multiply by $\boldsymbol{E}_{32}^{-1}$, $\boldsymbol{E}_{31}^{-1}$ and $\boldsymbol{E}_{21}^{-1}$ to factor $\boldsymbol{A}$ into $\boldsymbol{A} = \boldsymbol{LU}$, where $\boldsymbol{L} = \boldsymbol{E}_{21}^{-1}\boldsymbol{E}_{31}^{-1}\boldsymbol{E}_{32}^{-1}$ is a lower triangular matrix whose entries are the *multipliers* of elimination and $\boldsymbol{U}$ is an upper triangular matrix with the pivots on the diagonal.

(c) Factor $\boldsymbol{U} = \boldsymbol{D}\boldsymbol{U}_1$, where $\boldsymbol{D}$ is a diagonal matrix that contains the pivots.

(d) Factor $\boldsymbol{A} = \boldsymbol{LD}\boldsymbol{U}_1$, where $\boldsymbol{L}$ is a lower triangular matrix with 1s on the diagonal and whose other entries are the multipliers, $\boldsymbol{D}$ is a diagonal matrix that contains the pivots, and $\boldsymbol{U}_1$ is an upper triangular matrix with 1s on the diagonal.

6. Let $\boldsymbol{A}$ be a 3×3 matrix. Suppose you want to do the following operations on $\boldsymbol{A}$: Subtract row 1 from row 2, subtract row 1 from row 3, and subtract row 2 from row 3.

   (a) First, suppose you want to perform these operations on $\boldsymbol{A}$ *simultaneously*. Write down a matrix $\boldsymbol{B}$ that you need to multiply $\boldsymbol{A}$ by to achieve this. Does $\boldsymbol{AB}$ or $\boldsymbol{BA}$ yield the desired result? What is $\boldsymbol{B}^{-1}$? [*Hint*: You should be able to find $\boldsymbol{B}^{-1}$ by inspection, but it's not quite a simple as flipping the signs of the entries below the diagonal.]

   (b) Now, suppose you want to perform these operations on $\boldsymbol{A}$ *sequentially*. Write down a matrix $\boldsymbol{E}$ that you need to multiply $\boldsymbol{A}$ by to achieve this. It should be the product of three elementary matrices [*Hint*: It should *not* the same matrix as B from Part (a)]. What is $\boldsymbol{E}^{-1}$.

   (c) Suppose you factor $\boldsymbol{A} = \boldsymbol{LU}$, where $\boldsymbol{U}$ is the upper triangular matrix with the pivots on the diagonal, and $\boldsymbol{L}$ is lower-triangular. What is the relationship between $\boldsymbol{L}$ and $\boldsymbol{E}$?

7. A matrix is *symmetric* if $\boldsymbol{A}^T = \boldsymbol{A}$. Factor the following matrices into $\boldsymbol{A} = \boldsymbol{LDL}^T$.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}, \qquad \boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

   Note that for any symmetric matrix, the $\boldsymbol{A} = \boldsymbol{LDU}$ factorization simplifies to $\boldsymbol{A} = \boldsymbol{LDL}^T$.