Read: Strang, Section 3.1, 3.2.

Suggested short conceptual exercises: Strang, Section 3.1, #15, 16, 23, 25–27, 29. Section 3.2, #3, 6, 9, 10–18, 20, 28–31.

1. For each subsets of $\mathbb{R}^3$, determine if it is a subspace. If not, give an explicit example of how it fails.
   (a) The plane of vectors $\boldsymbol{b} = (b_1, b_2, b_3)$ with $b_1 = b_2$.
   (b) The plane of vectors with $b_1 = 1$.
   (c) The vectors with $b_1 b_2 b_3 = 0$.
   (d) All linear combinations of $\boldsymbol{v} = (1, 4, 0)$ and $\boldsymbol{w} = (2, 2, 2)$.
   (e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.
   (f) All vectors with $b_1 \leq b_2 \leq b_3$.

2. Let P be the plane in $\mathbb{R}^3$ with equation $x + y - 2z = 4$, which does not contain the zero-vector, and let P_0 be the parallel plane that passes through the origin.
   (a) Write the plane P_0 as an equation $ax + by + cz = d$.
   (b) Find two vectors in P and check that their sum is not in P.
   (c) Find two non-colinear vectors in P_0 and check that their sum is in P_0.
   (d) Write the set of vectors in P_0 as a linear combination $c\boldsymbol{v} + d\boldsymbol{w}$ for some $\boldsymbol{v}$ and $\boldsymbol{w}$.
   (e) Write the set of vectors in P as $\boldsymbol{x}_p + c\boldsymbol{v} + d\boldsymbol{w}$. That is, find such an $\boldsymbol{x}_p$ that works.
   (f) Find a 3×2 matrix $\boldsymbol{A}$ and a vector $\boldsymbol{b}$ such that the column space of $\boldsymbol{A}$ is the plane P_0 and the solutions to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is the plane P. [*Hint*: Any $\boldsymbol{x}_p$ that solves $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ will work!]

3. The matrix $\boldsymbol{A} = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ can be thought of as a "vector" is the space $\mathcal{M}$ of all 2×2 matrices.
   (a) Write down the zero vector in this space, the vector $\frac{1}{2}\boldsymbol{A}$, and the vector $-\boldsymbol{A}$.
   (b) What matrices are in the smallest subspace containing $\boldsymbol{A}$?
   (c) Describe a subspace of M that contains $\boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but not $\boldsymbol{C} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.
   (d) If a subspace of $\mathcal{M}$ contains $\boldsymbol{B}$ and $\boldsymbol{C}$, must it contain $\boldsymbol{I}$?
   (e) Describe a nontrivial subspace of $\mathcal{M}$ that contains no nonzero diagonal matrices.

4. Fix n, and let $\mathcal{M}$ be the space of all $n \times n$ matrices. For either of the following subsets of $\mathcal{M}$, determine whether it is a subspace. If true, give a reason, and if false, show why it fails with an explicit example.

(a) The set of invertible matrices.

(b) The set of singular (non-invertible) matrices.

(c) The set of symmetric matrices ($\boldsymbol{A}^T = \boldsymbol{A}$).

(d) The set of skew-symmetric matrices ($\boldsymbol{A}^T = -\boldsymbol{A}$).

(e) The set of unsymmetric matrices ($\boldsymbol{A}^T \neq \boldsymbol{A}$).

5. For each system, determine which right sides (find a condition on b_1, b_2, and b_3) that makes them solvable. Describe the column space geometrically (line, plane, $\mathbb{R}^3$, etc.) in each case.

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

6. Consider the following two matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$$

For each of these matrices, carry out the steps outlined below.

(a) Reduce the matrix to its ordinary echelon form $\boldsymbol{U}$. Determine which are the free variables and which are the pivot variables. What is its rank?

(b) Find a *special solution* for each free variable. That is, set the free variable to 1 and the other free variables to 0.

(c) By combining the special solutions, describe every solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{B}\boldsymbol{x} = \boldsymbol{0}$. This is the *nullspace* of the matrix.

(d) Write down the *nullspace matrix* $\boldsymbol{N}$, whose columns are the special solutions.

(e) By further row operations on each $\boldsymbol{U}$, find the reduced row echelon form $\boldsymbol{R}$.

(f) Embedded within the nonzero rows of $\mathbb{R}$ are two (not necessarily contiguous) submatrices, one an identity matrix and the other a matrix $\boldsymbol{F}$. Identify $\boldsymbol{F}$. Now, $\boldsymbol{N}$ consists of two (not necessarily contiguous) submatrices: what are they?

(g) True or false: The nullspace of $\boldsymbol{R}$ equals the nullspace of $\boldsymbol{U}$?

7. For each of the following part, construct a matrix $\boldsymbol{A}$ with the specific properties specified. In this problem, $n \times 1$ column vectors are written as ordered n-tuples.

(a) The column space contains $(1, 1, 1)$ and the nullspace is the line of multiples of $(1, 1, 1, 1)$.

(b) $\boldsymbol{A}$ is a 2×2 matrix whose column space equals its nullspace.

(c) The nullspace of $\boldsymbol{A}$ consists of all linear combinations of $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$. [*Hint*: Let these be the "special solutions" of a matrix $\boldsymbol{U}$ where x_3 and x_4 are the free varibles.]

(d) The nullspace of $\boldsymbol{A}$ consists of all multiples of $(4, 3, 2, 1)$. [*Hint*: Let these be the special solutions of a matrix $\boldsymbol{U}$ where x_4 is the only free varible.]

(e) The column space contains $(1, 1, 5)$ and $(0, 3, 1)$ and the nullspace contains $(1, 1, 2)$.