Read: Strang, Section 3.3, 3.4, 3.5.

Suggested short conceptual exercises: Strang, Section 3.3, #1, 8, 9, 11, 13, 16, 17–22. Section 3.4, #7, 13–17, 22, 24, 25, 27, 33. Section 3.5, #4, 9.

1. Find the reduced row echelon forms $\boldsymbol{R}$ and the rank of these matrices:
   (a) The 3×4 matrix with all entries equal to 2.
   (b) The 3×4 matrix with $a_{ij} = i + j - 1$.
   (c) The 3×4 matrix with $a_{ij} = (-1)^j$.
   (d) The transposes of each of the three matrices above.

2. What are the "special solutions" to $\boldsymbol{Rx} = \boldsymbol{0}$ for the matrices $\boldsymbol{R}$ shown below, and what is their rank?

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{R} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3. Consider the system $\boldsymbol{Ax} = \boldsymbol{b}$, where

$$\boldsymbol{A} = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}.$$

   Solve this system by carrying out the following steps.
   (a) Reduce $[\boldsymbol{A} \;\; \boldsymbol{b}]$ to $[\boldsymbol{U} \;\; \boldsymbol{c}]$, turning the system $\boldsymbol{Ax} = \boldsymbol{b}$ into an upper triangular one, $\boldsymbol{Ux} = \boldsymbol{c}$.
   (b) Find condition(s) on b_1, b_2, and b_3 for $\boldsymbol{Ax} = \boldsymbol{b}$ to have a solution. Each "zero row" of U will give you a condition.
   (c) Describe the *column space* $C(\boldsymbol{A})$ of $\boldsymbol{A}$ as a subspace of $\boldsymbol{R}^3$. Express it as a linear combination of a *minimal* number of vectors.
   (d) Describe the *nullspace* $\boldsymbol{x}_n := N(\boldsymbol{A})$ of $\boldsymbol{A}$. What are the special solutions that generate it?
   (e) Find any *particular solution* to $\boldsymbol{Ax} = \boldsymbol{b}$ and then the *general solution*, which will have the form $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n$.
   (f) Reduce $[\boldsymbol{U} \;\; \boldsymbol{c}]$ to $[\boldsymbol{R} \;\; \boldsymbol{d}]$, where $\boldsymbol{R}$ is the row-reduced echelon form and $\boldsymbol{d}$ is a particular solution.

4. Carry out the steps in the previous problem for the following system.

$$\boldsymbol{A} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ 4 & 2 & 6 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 10 \\ 30 \\ 20 \end{bmatrix}.$$

   Aside from being only 3×3, this matrix is only of rank 1, so there will be a few noticable differences.

5. Consider the following 3×3 matrices.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}.$$

Carry out the following steps for both $\boldsymbol{A}$ and $\boldsymbol{B}$.

(a) Determine which vectors (b_1, b_2, b_3) are in the column space. [*Hint*: Each "zero row" should give you a condition on b_1, b_2, b_3.]

(b) What combination of the rows give the zero row?

(c) What is the relationship between Parts (a) and (b)?

6. Suppose we have a system $\boldsymbol{Ax} = \boldsymbol{b}$ with particular solution $\boldsymbol{x}_p = (2, 4, 0)$ and whose "homogeneous" solution $\boldsymbol{x}_n$ (i.e., the nullspace of A) is the set of scalar multiples of $(1, 1, 1)$.

(a) Construct such a 2×3 system. Sketch the "grid picture."

(b) Why can't there be a 1×3 system satisfying these conditions? Sketch the "grid picture" and show how it fails.

7. Find matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ with the given property or explain why there is none. [*Hint*: Recall that $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n$. Sketching the "grid picture" isn't necessary, but it may help.]

(a) The only solution of $\boldsymbol{Ax} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(b) The only solution of $\boldsymbol{Bx} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

8. Consider the following four vectors:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

(a) Show that $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ are linearly independent but $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, $\boldsymbol{v}_4$ are linearly dependent.

(b) Solve $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 + c_4\boldsymbol{v}_4 = \boldsymbol{0}$, or alternatively, $\boldsymbol{Ax} = \boldsymbol{0}$ where the $\boldsymbol{v}_i$'s are the columns of $\boldsymbol{A}$.

9. Let P be the hyperplane $x + 2y - 3z - t = 0$ in $\boldsymbol{R}^4$.

(a) Find two linearly independent vectors on P.

(b) Find three linearly independent vectors on P.

(c) Why can you not find four linearly independent vectors on P?

(d) Find a matrix A whose column space is P, and a matrix B whose nullspace is P.