Read: Strang, Section 3.5, 3.6.

Suggested short conceptual exercises: Strang, Section 3.5, #11, 12, 14, 15, 18, 19, 21, 22, 24. Section 3.6, #1, 7, 12, 13, 15, 16, 24–26.

Throughout, "the four subspaces" associated with a matrix $\boldsymbol{A}$ refer to the column space $C(\boldsymbol{A})$, row space $C(\boldsymbol{A}^T)$, nullspace $N(\boldsymbol{A})$, and left nullspace $N(\boldsymbol{A}^T)$.

1. Let $\mathcal{M}$ be the space of all 2×3 matrices.
   (a) Find a basis for the subspace S consisting of the matrices in $\mathcal{M}$ whose columns add to zero.
   (b) Find a basis for the subspace T consisting of the matrices in $\mathcal{M}$ whose rows add to zero.
   (c) Find a basis for the subspace $S \cap T$ of $\mathcal{M}$. This is the set of matrices whose rows and columns add to zero.
   (d) Describe the subspace $S + T$, which is the set of all sums of matrices in S with matrices in T.
   (e) Compute $\dim(S + T)$, and express it in terms of $\dim S$, $\dim T$, and $\dim(S \cap T)$.
   (f) Find a basis of the subspace U of $\mathcal{M}$ whose nullspace contains $(2, 1, 1)$.
2. Let $\mathcal{P}_3$ be the set of polynomials of degree at most 3, which is a vector space.
   (a) Determine a basis and the dimension of $\mathcal{P}_3$.
   (b) Explain why the set of polynomials of degree exactly 3 is not a vector space.
   (c) The set of polynomials satifying $p''(x) = 0$ is a subspace of $\mathcal{P}_3$. Find a basis for it and its dimension.
   (d) Find a basis for the subspace of $\mathcal{P}_3$ consisting of the polynomials with $p(1) = 0$.
3. The set $\mathcal{C}^\infty$ of smooth (i.e., infinitely differentiable) functions is an infinite dimensional vector space. Let D be the differential operator d/dx, which is a *linear map* from $\mathcal{C}^\infty$ to itself. (Being *linear* just means that $D(af + bg) = aD(f) + bD(g)$ holds for all constants a and b, and functions f and g.)
   (a) What is the *nullspace* of D? Write down an explicit basis and its dimension.
   (b) Find a particular function $y_p(x)$ such that $Dy_p = 3$.
   (c) Use Parts (a) and (b) to find *all* functions that satisfy $Dy = 3$.
4. Let D be the differential operator $d/dx - 1$, which is a *linear map* from the space of $\mathcal{C}^\infty$ of smooth functions to itself. Explicitly, if $y(x)$ is a function, then

$$D\colon y(x) \longmapsto \left(\frac{d}{dx} - 1\right) y(x) = y'(x) - y(x)\,.$$

(a) The nullspace of D is one-dimensional (you may assume this – it is a well-known fact that an n^{th} order linear differential operator has an n-dimensional nullspace). Write down an explicit basis for the nullspace of D. [*Hint*: You should be able to do this by inspection using only basic differential calculus.]

(b) Find a particular function $y_p(x)$ such that $Dy_p = 3$, i.e., *any* function satifying the equation $y'(x) - y(x) = 3$. [*Hint*: Try a constant function, $y_p(x) = c$. What c works?]

(c) Use Parts (a) and (b) to find *all* functions that satisfy $Dy = 3$. [This is the *general solution* of the *differential equation* $y' = y + 3$. It's a vector space!]

5. Find bases and the dimensions of the four subspaces associated with A and B.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 7 \end{bmatrix}.$$

6. Consider the matrix $\boldsymbol{A}$, whose $\boldsymbol{LU}$-factorization is given below. Find a basis for and compute the dimension of each of the four subspaces associated with $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 7 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \boldsymbol{LU}.$$

7. Without using elimination, find bases for the four subspaces of the following matrices

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

8. Let P be the plane spanned by $(1, 1, 1)$ and $(1, 2, 0)$.

(a) Find a matrix that has P as its column space.

(b) Find a matrix that has P as its row space.

(c) Find a matrix that has P as its nullspace.

(d) Find a matrix that has P as its left nullspace.

9. Let $\boldsymbol{A}$ be an $m \times n$ matrix of rank r. Suppose there are n-dimensional vectors $\boldsymbol{b}$ for which $\boldsymbol{Ax} = \boldsymbol{b}$ has no solution.

(a) List all inequalities ($<$ or $\leq$) that must hold between m, n, and r.

(b) Explain why $\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}$ must have a non-zero solution.