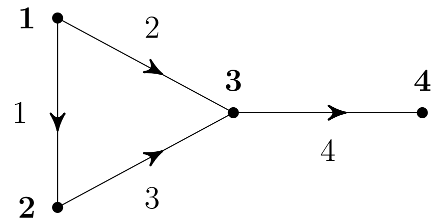
Read: Strang, Section 8.2. *Suggested short conceptual exercises*: #15–18.

Consider the graph shown below, on $n = 4$ vertices and $m = 4$ edges.



Answer the following questions. *Write clearly and concisely*! Part of your grade will be on the presentation of your solutions.

1. Write down the 4×4 incidence matrix $\boldsymbol{A}$ for this graph, given the vertex and edge labels above. Compute the rank of $\boldsymbol{A}$ and solve $\boldsymbol{Ax} = \boldsymbol{0}$, thereby finding its nullspace. The components of $\boldsymbol{x}$ are the *potentials* at the vertices. Describe in a sentence what a vector $\boldsymbol{x}$ in $N(\boldsymbol{A})$ physically represents. What type of graph would have $N(\boldsymbol{A})$ of dimension greater than 1?

2. Solve $\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}$ for $\boldsymbol{y}$, thereby finding the left nullspace of $\boldsymbol{A}$. Recall that $\operatorname{rank} \boldsymbol{A} = \operatorname{rank} \boldsymbol{A}^T$. The components of $\boldsymbol{y}$ are *currents* on the edges. Describe in a sentence what a vector $\boldsymbol{y}$ in $N(\boldsymbol{A}^T)$ physically represents.

3. Use elimination to find the ordinary echelon matrix $\boldsymbol{U}$ of $\boldsymbol{A}$. What *spanning tree* corresponds to the nonzero rows of $\boldsymbol{U}$? Sketch this tree on the network.

4. Determine the requirement(s) that the b_i's must satisfy for $\boldsymbol{Ax} = \boldsymbol{b}$ to have a solution. The physical interpretation of this is *Kirchoff's voltage law* (KVL) – the components of $\boldsymbol{Ax}$ add to zero around every loop (so no voltage drop). How are the feasible $\boldsymbol{b}$'s related to $\boldsymbol{y} = (1, -1, 1, 0)$? Decribe in a sentence what a vector $\boldsymbol{x}$ solving $\boldsymbol{Ax} = \boldsymbol{b}$ physically represents.

5. The equation $\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{f}$ is *Kirchoff's current law* (KCL), and physically represents that at each vertex, flow in equals flow out (including sources). Choose a nonzero vector (f_1, f_2, f_3, f_4) for which $\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{f}$ can be solved, and solve for $\boldsymbol{y}$. For this $\boldsymbol{y}$, sketch the physical situation of the potential, current flows, and voltage sources and sink on the network. How are the feasible $\boldsymbol{f}$'s related to $\boldsymbol{x} = (1, 1, 1, 1)$?

Ohm's Law states that current is $\boldsymbol{y} = -\boldsymbol{CAx}$, where the $\boldsymbol{C}$ is the *conductance matrix* which is diagonal where the (i, i)-entry is th conductance c_i on edge i. The relationship between potentials, currents, and voltage sources and sinks is summarized in the diagram below.

$$\begin{tikzcd}
\text{node potentials} & \boldsymbol{x} \arrow[rr, dashed, "\boldsymbol{A}^T\boldsymbol{CA}"] \arrow[d, "\boldsymbol{A}"'] & & \boldsymbol{A}^T\boldsymbol{CAx} = \boldsymbol{f} & \text{KCL} \\
\text{potential differences; KVL} & \boldsymbol{Ax} \arrow[rr, "\boldsymbol{C}", "\text{Ohm's law}"'] & & \boldsymbol{y} = \boldsymbol{CAx} \arrow[u, "\boldsymbol{A}^T"'] & \text{(neg.) currents}
\end{tikzcd}$$

Remark. In circuit theory (in physics and electrical engineering), current flows from higher potential to lower potential, which is why current is $\boldsymbol{y} = -\boldsymbol{C}\boldsymbol{A}\boldsymbol{x}$, rather than $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{A}\boldsymbol{x}$. This artificial negative sign does not appear when this equation arises in mechanical engineering, where $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{A}\boldsymbol{x}$ is called *Hooke's Law* for mass-spring systems, and $\boldsymbol{C}$ is the diagonal matrix of the spring elasticities.

6. First, consider the case when all conductances are 1, so $\boldsymbol{C} = \boldsymbol{I}$, and compute $\boldsymbol{A}^T\boldsymbol{C}\boldsymbol{A} = \boldsymbol{A}^T\boldsymbol{A}$ (recall that it is symmetric!). Next choose a vector $\boldsymbol{x}$ with non-negative entries (because potentials are non-negative) and then compute the resulting currents $\boldsymbol{y} = -\boldsymbol{C}\boldsymbol{A}\boldsymbol{x}$ and the vector $\boldsymbol{f}$ of sources and sinks. Note that KCL dictates that $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{f}$ must hold. Sketch the network and include the potentials $\boldsymbol{x}$, the currents $\boldsymbol{y}$, and the sources and sinks $\boldsymbol{f}$ on the graph. Remember that at each node, flow in must equal flow out!

7. Now, suppose the conductances are $c_1 = 1$ and $c_2 = c_3 = c_4 = 2$. Moreover, let $\boldsymbol{f} = (8, 0, -8, 0)$ be the vector of sources and sinks. Determine the potentials at each node by solving $\boldsymbol{A}^T\boldsymbol{C}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{f}$ for $\boldsymbol{x}$. Sketch the network and include the potentials $\boldsymbol{x}$, the currents $\boldsymbol{y}$, and the sources and sinks $\boldsymbol{f}$.