Read: Strang, Section 6.1, 6.2.

Suggested short conceptual exercises: Strang, Section 6.1, #1, 7, 8, 13, 16, 17, 19, 23, 25, 29. Section 6.2, #3–5, 7, 11–14, 20–22, 24, 25, 29, 30, 32, 34.

1. Consider the following matrices:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \qquad \boldsymbol{A}^2 = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}, \qquad \boldsymbol{A} + \boldsymbol{I} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \qquad \boldsymbol{A}^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

   (a) Find the eigenvalues and eigenvectors of $\boldsymbol{A}$.

   (b) Diagonalize $\boldsymbol{A}$, by writing $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1}$, where $\boldsymbol{\Lambda}$ is a diagonal matrix of eigenvalues and $\boldsymbol{S}$ is the matrix of corresponding eigenvectors. (No need to actually compute $\boldsymbol{S}^{-1}$ by taking the inverse; just put a -1 exponent on $\boldsymbol{S}$.)

   (c) Repeat the previous parts for the matrices $\boldsymbol{A}^2$, $\boldsymbol{A}+\boldsymbol{I}$, and $\boldsymbol{A}^{-1}$.

2. Find the eigenvalues of the following matrices:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{AB} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \qquad \boldsymbol{BA} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

   (a) Are the eigenvalues of $\boldsymbol{AB}$ equal to the product of the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$?

   (b) Do $\boldsymbol{AB}$ and $\boldsymbol{BA}$ have the same eigenvalues?

3. Suppose λ is an eigenvalue of $\boldsymbol{A}$. Prove the following:

   (a) λ^2 is an eigenvalue of $\boldsymbol{A}^2$.

   (b) If $\boldsymbol{A}$ is invertible, then λ^{-1} is an eigenvalue of $\boldsymbol{A}^{-1}$.

   (c) $\lambda + c$ is an eigenvalue of $\boldsymbol{A} + c\boldsymbol{I}$, where c is a constant.

   (d) $\boldsymbol{A}$ and $\boldsymbol{A}^T$ have the same eigenvalues.

4. The *characteristic polynomial* of $\boldsymbol{A}$ is $\chi_{\boldsymbol{A}}(\lambda) = \det(\boldsymbol{A} - \lambda\boldsymbol{I})$. Suppose this factors (always possible) as

$$\chi_{\boldsymbol{A}}(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda).$$

   (a) Plug in $\lambda = 0$ and find a formula for $\det \boldsymbol{A}$ in terms of the eigenvalues of $\boldsymbol{A}$.

   (b) The *trace* of $\boldsymbol{A}$, denoted $\operatorname{tr}\boldsymbol{A}$, is the sum of the diagonal entries which is also (amazingly!) equal to the sum of the eigenvalues. If $\boldsymbol{A}$ is 2×2, then

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has} \quad \det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

   Write a formula for the chacteristic polynomial of a 2×2 matrix in terms of $\det \boldsymbol{A}$ and $\operatorname{tr}\boldsymbol{A}$.

   (c) Suppose $\boldsymbol{A}$ is an $n \times n$ matrix with characteristic polynomial $\chi_{\boldsymbol{A}} = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$. Describe $\det \boldsymbol{A}$ and $\operatorname{tr}\boldsymbol{A}$ in terms of the c_i's.

(d) Explain why $\boldsymbol{AB} - \boldsymbol{BA} = \boldsymbol{I}$ is impossible for $n \times n$ matrices.

5. Find the eigenvalues and eigenvectors of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$. Compute the traces and determinants of these matrices as well.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

6. Recall that the following matrix rotates the xy-plane by the angle θ:

$$\boldsymbol{Q} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

(a) Solve $\det(\boldsymbol{Q} - \lambda \boldsymbol{I}) = 0$ by the quadratic formula to find the eigenvalues of $\boldsymbol{Q}$.

(b) Find the eigenvectors of $\boldsymbol{Q}$ by solving $(\boldsymbol{Q} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$. Use $i^2 = -1$.

7. Suppose $\boldsymbol{A}$ has eigenvalues 0, 3, 5 with linearly independent eigenvectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$.

(a) Give a basis for the nullspace and a basis for the column space.

(b) Find a particular solution to $\boldsymbol{Ax} = \boldsymbol{v} + \boldsymbol{w}$. Find all solutions.

(c) Explain why $\boldsymbol{Ax} = \boldsymbol{u}$ has no solution.

8. If $\boldsymbol{A}$ has $\lambda_1 = 2$ with eigenvector $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 5$ with $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, use $\boldsymbol{S\Lambda S}^{-1}$ to find $\boldsymbol{A}$. Note that no other matrix has the same λ's and $\boldsymbol{x}$'s!

9. Consider the following matrices

$$\boldsymbol{A}_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix}, \qquad \boldsymbol{A}_2 = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix}, \qquad \boldsymbol{A}_3 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

(a) Diagonalize each matrix by writing $\boldsymbol{A} = \boldsymbol{S\Lambda S}^{-1}$.

(b) For each of these three matrices, compute the limit $\boldsymbol{A}^k = \boldsymbol{S\Lambda}^k\boldsymbol{S}^{-1}$ as $k \to \infty$ if it exists.

(c) Suppose $\boldsymbol{A}$ is an $n \times n$ matrix that is diagonalizable (so it has n linearly independent eigenvectors). What must be true for the limit $\boldsymbol{A}^k$ to exist as $k \to \infty$? What is needed for $\boldsymbol{A}^k \to \boldsymbol{0}$? Justify your answer.

(d) Compute $(\boldsymbol{A}_3)^{10}\boldsymbol{u}_0$ for the following $\boldsymbol{u}_0$:

$$\boldsymbol{u}_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \boldsymbol{u}_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad \boldsymbol{u}_0 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$