Read: Strang, Section 6.1, 6.2.


1. Consider the following matrices:
   \[A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}, \quad A + I = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.\]
   (a) Find the eigenvalues and eigenvectors of \(A\).
   (b) Diagonalize \(A\), by writing \(A = S\Lambda S^{-1}\), where \(\Lambda\) is a diagonal matrix of eigenvalues and \(S\) is the matrix of corresponding eigenvectors. (No need to actually compute \(S^{-1}\) by taking the inverse; just put a \(-1\) exponent on \(S\).)
   (c) Repeat the previous parts for the matrices \(A^2\), \(A + I\), and \(A^{-1}\).

2. Find the eigenvalues of the following matrices:
   \[A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.\]
   (a) Are the eigenvalues of \(AB\) equal to the product of the eigenvalues of \(A\) and \(B\)?
   (b) Do \(AB\) and \(BA\) have the same eigenvalues?

3. Suppose \(\lambda\) is an eigenvalue of \(A\). Prove the following:
   (a) \(\lambda^2\) is an eigenvalue of \(A^2\).
   (b) If \(A\) is invertible, then \(\lambda^{-1}\) is an eigenvalue of \(A^{-1}\).
   (c) \(\lambda + c\) is an eigenvalue of \(A + cI\), where \(c\) is a constant.
   (d) \(A\) and \(A^T\) have the same eigenvalues.

4. The characteristic polynomial of \(A\) is \(\chi_A(\lambda) = \det(A - \lambda I)\). Suppose this factors (always possible) as
   \[\chi_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda).\]
   (a) Plug in \(\lambda = 0\) and find a formula for \(\det A\) in terms of the eigenvalues of \(A\).
   (b) The trace of \(A\), denoted \(\text{tr} A\), is the sum of the diagonal entries which is also (amazingly!) equal to the sum of the eigenvalues. If \(A\) is \(2 \times 2\), then
   \[A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\]
   has \(\det(A - \lambda I) = \lambda^2 - (a + d)\lambda + (ad - bc) = 0\).

   Write a formula for the characteristic polynomial of a \(2 \times 2\) matrix in terms of \(\det A\) and \(\text{tr} A\).
   (c) Suppose \(A\) is an \(n \times n\) matrix with characteristic polynomial \(\chi_A = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0\). Describe \(\det A\) and \(\text{tr} A\) in terms of the \(c_i\)'s.
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(d) Explain why $\mathbf{AB} - \mathbf{BA} = \mathbf{I}$ is impossible for $n \times n$ matrices.

5. Find the eigenvalues and eigenvectors of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. Compute the traces and determinants of these matrices as well.

$$
\mathbf{A} = \begin{bmatrix}
2 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}, \quad 
\mathbf{B} = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}, \quad 
\mathbf{C} = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}.
$$

6. Recall that the following matrix rotates the $xy$-plane by the angle $\theta$:

$$
\mathbf{Q} = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \\
\end{bmatrix}.
$$

(a) Solve $\det(\mathbf{Q} - \lambda \mathbf{I}) = 0$ by the quadratic formula to find the eigenvalues of $\mathbf{Q}$.

(b) Find the eigenvectors of $\mathbf{Q}$ by solving $(\mathbf{Q} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$. Use $i^2 = -1$.

7. Suppose $\mathbf{A}$ has eigenvalues 0, 3, 5 with linearly independent eigenvectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

(a) Give a basis for the nullspace and a basis for the column space.

(b) Find a particular solution to $\mathbf{Ax} = \mathbf{v} + \mathbf{w}$. Find all solutions.

(c) Explain why $\mathbf{Ax} = \mathbf{u}$ has no solution.

8. If $\mathbf{A}$ has $\lambda_1 = 2$ with eigenvector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 5$ with $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, use $\mathbf{SAS}^{-1}$ to find $\mathbf{A}$. Note that no other matrix has the same $\lambda$’s and $\mathbf{x}$’s!

9. Consider the following matrices

$$
\mathbf{A}_1 = \begin{bmatrix}
.6 & .9 \\
.4 & .1 \\
\end{bmatrix}, \quad 
\mathbf{A}_2 = \begin{bmatrix}
.6 & .2 \\
.4 & .8 \\
\end{bmatrix}, \quad 
\mathbf{A}_3 = \begin{bmatrix}
.6 & .9 \\
.1 & .6 \\
\end{bmatrix}.
$$

(a) Diagonalize each matrix by writing $\mathbf{A} = \mathbf{SAS}^{-1}$.

(b) For each of these three matrices, compute the limit $\mathbf{A}^k = \mathbf{S}\Lambda^k\mathbf{S}^{-1}$ as $k \to \infty$ if it exists.

(c) Suppose $\mathbf{A}$ is an $n \times n$ matrix that is diagonalizable (so it has $n$ linearly independent eigenvectors). What must be true for the limit $\mathbf{A}^k$ to exist as $k \to \infty$? What is needed for $\mathbf{A}^k \to \mathbf{0}$? Justify your answer.

(d) Compute $(\mathbf{A}_3)^{10}\mathbf{u}_0$ for the following $\mathbf{u}_0$:

$$
\mathbf{u}_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad 
\mathbf{u}_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad 
\mathbf{u}_0 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.
$$