Read: Strang, Section 6.4, 8.3, 8.5, 10.1, 10.2.

Suggested short conceptual exercises: Strang, Section 6.4, #2, 7–10, 13–15, 18, 20–22, 25, 26. Section 8.3, #1, 4, 6, 8–11, 17. Section 8.5, #5, 10.

1. Consider the following three Markov matrices.

$$\boldsymbol{A} = \begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix} \qquad \boldsymbol{A} = \begin{bmatrix} .2 & 1 \\ .8 & 0 \end{bmatrix} \qquad \boldsymbol{A} = \begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}.$$

   Carry out the following steps for each matrix.

   (a) Draw a weighted directed graph on n (=2 or 3) vertices showing the transition probabilities between states.

   (b) Find the eigenvalues and steady-state eigenvector.

   (c) Diagonalize $\boldsymbol{A}$ by writing $\boldsymbol{A} = \boldsymbol{S\Lambda S}^{-1}$ and compute $\lim_{k\to\infty} \boldsymbol{A}^k$.

2. Every year, 10% of young people become old, 0.1% of young people become dead, and 5% of old people become dead. Assume there are no births and no zombies.

   (a) Find the Markov matrix $\boldsymbol{A}$ that models this process:

$$\begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_{k+1} = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_{k}.$$

   (b) Find the steady-state eigenvector by inspection alone.

   (c) Find the eigenvalues and eigenvectors of $\boldsymbol{A}$. (Use a computer.)

   (d) Suppose there are 1000 young people initially. Compute the number of young, old, and dead people after 10 years. (Again, use a computer, but also use $\boldsymbol{A}^k = \boldsymbol{S\Lambda}^k\boldsymbol{S}^{-1}$.)

3. Consider the set of all complex-valued 2π-periodic functions, which means that $f(z+2\pi) = f(z)$ for all z. This is a vector space and the infinite set $\{e^{inx} : n \in \mathbb{Z}\}$ is a basis. Define an *inner product* (i.e., dot product) on this space by

$$\langle f, g\rangle = \int_0^{2\pi} f(x)\overline{g(x)}\,dx.$$

   (a) Compute $\langle e^{inx}, e^{imx}\rangle$ and verify that this basis is indeed orthogonal. Recall that $\overline{e^{ix}} = e^{-ix}$, and be sure to consider the cases when $n = m$ and $n \neq m$ separately.

   (b) Since $\{e^{inx} : n \in \mathbb{Z}\}$ is a basis, we can write any 2π-periodic function $f(x)$ as

$$f(x) = \quad \cdots + c_{-2}e^{-2ix} + c_{-1}e^{-ix} + c_0 + c_1e^{ix} + c_2e^{2ix} + \cdots \quad = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

   Derive a formula for c_n. [*Hint*: Right-multiply both sides of the above equation by $\overline{e^{inx}}$ and integrate.]

(c) Give an *orthonormal basis* for this vector space.

(d) Define the function $f(x) = e^x$ on the interval $[0, 2\pi]$, and extend f to be periodic. Sketch the graph of this function and compute its complex Fourier series. That is, compute the coefficients c_n. You'll need to compute c_n (for $n \neq 0$) and c_0 separately.

4. Let $\boldsymbol{A}$ be an $n \times n$ *Hermitian* matrix (also called *self-adjoint*), which means that $\overline{\boldsymbol{A}}^T = \boldsymbol{A}$. Note that $\boldsymbol{A}$ may have complex-valued entries.

   (a) Which Hermitian matrices are also symmetric?

   (b) Prove that all eigenvalues of $\boldsymbol{A}$ are real.

   (c) Prove that if $\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$ and $\boldsymbol{A}\boldsymbol{w} = \mu \boldsymbol{w}$ for $\lambda \neq \mu$, then $\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal.

   These two proofs can be done by modifying the arguments in class that proved these statements for real symmetric matrices.

5. An real-valued $n \times n$ matrix is *skew-symmetric* if $\boldsymbol{A}^T = -\boldsymbol{A}$.

   (a) What must be true about the diagonal entries of a skew-symmetric matrix?

   (b) Prove that every non-zero eigenvalue of a skew-symmetric matrix is purely imaginary. [*Hint*: Note that a complex eigenvalue $\lambda = a + bi$ is real if $\lambda = \overline{\lambda}$, whereas it is purely imaginary if $\lambda = -\overline{\lambda}$.]

   (c) Prove that if n is odd and $\boldsymbol{A}$ is skew-symmetric, then $\boldsymbol{A}$ must be singular. [*Hint*: complex eigenvalues come in pairs, because $\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$ implies $\overline{\boldsymbol{A}\boldsymbol{v}} = \boldsymbol{A}\overline{\boldsymbol{v}} = \overline{\lambda}\overline{\boldsymbol{v}}$.]

   (d) Write $\boldsymbol{A} = \boldsymbol{M} + \boldsymbol{N}$, the sum of a *symmetric* and a *skew-symmetric* matrix:

   $$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 5 \\ 4 & 3 & 0 \\ 7 & 6 & 5 \end{bmatrix} = \boldsymbol{M} + \boldsymbol{N} \qquad\qquad (\boldsymbol{M}^T = \boldsymbol{M}, \quad \boldsymbol{N}^T = -\boldsymbol{N}).$$

   Hint: For any square matrix, $\boldsymbol{M} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T)$. What does this force $\boldsymbol{N}$ to be?

6. Diagonalize the following matrices into $\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$, where $\boldsymbol{Q}$ is an *orthogonal* matrix.

   $$\boldsymbol{A} = \begin{bmatrix} 7 & 6 \\ 6 & -2 \end{bmatrix} \qquad\qquad \boldsymbol{B} = \begin{bmatrix} 0 & -2 & 2 \\ -2 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

   Find *all* orthogonal matrices that diagonalize $\boldsymbol{A}$. How many will diagonalize $\boldsymbol{B}$?

7. Which of these classes of matrices do $\boldsymbol{A}$ and $\boldsymbol{B}$ belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

   $$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad\qquad \boldsymbol{B} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

   Which of these factorizations are possible for $\boldsymbol{A}$ and $\boldsymbol{B}$: $\boldsymbol{L}\boldsymbol{U}$, $\boldsymbol{Q}\boldsymbol{R}$, $\boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1}$, $\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$?

8. What number b in $\boldsymbol{A} = \begin{bmatrix} 2 & b \\ 1 & 0 \end{bmatrix}$ makes $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$ possible? What number makes $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1}$ impossible? What number makes $\boldsymbol{A}^{-1}$ impossible?