Read: Strang, Section 6.5, 6.6, 10.1.

Suggested short conceptual exercises: Strang, Section 6.5, #1, 6, 10, 14–20, 27–31, 33. Section 6.6, #1, 4, 7, 8, 12–15, 17, 20.

1. Consider the following permutation matrix:

$$\boldsymbol{P}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

   (a) Find the four (complex-valued) eigenvalues and eigenvectors of $\boldsymbol{P}_4$.

   (b) Write out the four eigenvalues you just found in polar form, $\lambda = Re^{i\theta}$. Write the corresponding eigenvectors in polar form, normalized so the first entry is 1, i.e., $\boldsymbol{v}_k = (e^{i0}, -, -, -) = (1, -, -, -)$.

   (c) Without actually computing them, venture a guess as to what the eigenvalues and eigenvectors of $\boldsymbol{P}_6$ are – the 6×6 matrix with five 1's below the diagonal and one up the upper-right corner.

2. Consider the following two quadratic forms:

$$f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 5x_2^2, \qquad\qquad f(x_1, x_2) = 2x_1^2 + 6x_1x_2 + 2x_2^2.$$

   Carry out the following steps for each of these functions.

   (a) Write $f(x_1, x_2) = \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}$ for some $\boldsymbol{A}$.

   (b) Diagonalize $\boldsymbol{A}$ by writing $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$ for an orthogonal matrix $\boldsymbol{Q}$ that additionally has $\det \boldsymbol{Q} = +1$. [*Hint*: Every 2×2 orthogonal matrix with determinant 1 is a *rotation matrix* $\boldsymbol{Q} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for some $0 \leq \theta < 2\pi$. Find θ.]

   (c) Introduce new coordinates by setting $\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{y}$, and substitute $\boldsymbol{y} = \boldsymbol{Q}^T\boldsymbol{x}$ into the quadratic form $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}$. Write the result both as an equation $f(y_1, y_2)$, and in matrix form, $\boldsymbol{y}^T \boldsymbol{\Lambda} \boldsymbol{y}$.

   (d) Sketch the conic section $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} = 1$ on the x_1x_2-plane, and sketch $\boldsymbol{y}^T \boldsymbol{\Lambda} \boldsymbol{y} = 1$ on the y_1y_2-plane (okay to use a computer).

   (e) Write $f(x_1, x_2)$ as either the sum or the difference of two squares. (Only one will be possible.)

3. Consider the following quadratic form:

$$\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} & & \\ & \boldsymbol{A} & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2.$$

   Find the 3×3 matrix $\boldsymbol{A}$ and its pivots, rank, eigenvalues, and three (upper-left) subdeterminants:

4. Suppose $\boldsymbol{B} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}$ is an $n \times n$ matrix. The relationship between $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{M}$ as functions $\mathbb{R}^n \to \mathbb{R}^n$ is shown in the following *commutative diagram*:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "B"] \arrow[d, "M"'] & \mathbb{R}^n \arrow[d, "M"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \end{tikzcd}$$

Remember that matrix multiplication represents function composition, and so should be read from *right-to-left*.

(a) Draw a commutative diagram showing $\boldsymbol{B}^2 = (\boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M})(\boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}) = \boldsymbol{M}^{-1}\boldsymbol{A}^2\boldsymbol{M}$. [*Hint*: Imagine "stacking" two diagrams horizontally.]

(b) Suppose that $\boldsymbol{B} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}$ and $\boldsymbol{C} = \boldsymbol{N}^{-1}\boldsymbol{B}\boldsymbol{N}$. Draw a commutative diagram showing how $\boldsymbol{A}$ is similar to $\boldsymbol{C}$. Write this out algebraically as well. You have just proven that similarity is *transitive*.

(c) Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same eigenvalues $\lambda_1, \dots, \lambda_n$, all distinct. Prove that $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar.

(d) Show by example how the result in Part (c) fails if the eigenvalues are not distinct.

5. Show that each pair $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ are similar by finding $\boldsymbol{M}_i$ such that $\boldsymbol{B}_i = \boldsymbol{M}_i^{-1}\boldsymbol{A}_i\boldsymbol{M}_i$.

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \ \boldsymbol{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{A}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ \boldsymbol{B}_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \ \boldsymbol{A}_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \ \boldsymbol{B}_3 = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}.$$

6. There are sixteen 2×2 matrices whose entries are 0's and 1's. Partition these matrices into "families" (equivalence classes) where similar matrices go into the same family.

7. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ matrices with $\boldsymbol{B}$ invertible.

(a) Prove that $\boldsymbol{A}\boldsymbol{B}$ is similar to $\boldsymbol{B}\boldsymbol{A}$.

(b) Illustrate your proof from Part (a) by correctly labeling the six maps in the following commutative diagram:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r] \arrow[d] & \mathbb{R}^n \arrow[r] & \mathbb{R}^n \arrow[d] \\ \mathbb{R}^n \arrow[r] & \mathbb{R}^n \arrow[r] & \mathbb{R}^n \end{tikzcd}$$

(c) Conclude that $\boldsymbol{A}\boldsymbol{B}$ and $\boldsymbol{B}\boldsymbol{A}$ have the same eigenvalues.

8. Consider the following Jordan blocks with eigenvalue λ:

$$\boldsymbol{J}_2 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad \boldsymbol{J}_3 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \qquad \boldsymbol{J}_4 = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

For each $\boldsymbol{J}_i$, compute $\boldsymbol{J}_i^2$ and $\boldsymbol{J}_i^3$. Guess the form of $\boldsymbol{J}_i^k$. Set $k = 0$ to find $\boldsymbol{J}_i^0$ and $k = -1$ to find $\boldsymbol{J}_i^{-1}$