Read: Strang, Section 6.7, 7.1, 7.2, 7.3.

Suggested short conceptual exercises: Strang, Section 6.7, #3, 9–13, 15. Section 7.1, #1–12, 20–23, 29, 31. Section 7.2, #10, 13, 17, 26–29, 32.

1. Consider the following matrices: $\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$, $\quad \boldsymbol{A}^T\boldsymbol{A} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}$, $\quad \boldsymbol{A}\boldsymbol{A}^T = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$.
   (a) Find the eigenvalues σ_1^2, σ_2^2 and unit eigenvectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ of $\boldsymbol{A}^T\boldsymbol{A}$.
   (b) For the $\sigma_i \neq 0$, compute $\boldsymbol{u}_i = \boldsymbol{A}\boldsymbol{v}_i/\sigma_i$ and verify that indeed $||\boldsymbol{u}_i|| = 1$. Find the other $\boldsymbol{u}_i$ by computing the other unit eigenvector of $\boldsymbol{A}\boldsymbol{A}^T$.
   (c) Write out the singular value decomposition (SVD), $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$.
   (d) Write down orthonormal bases for the four fundamental subspaces of $\boldsymbol{A}$.
   (e) Describe *all* matrices that have the same four fundamental subspaces.

2. Which of these transformations satisfy $T(\boldsymbol{v}+\boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$ and which satisfy $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$? Assume that the mapping is either $\mathbb{R}^2 \to \mathbb{R}^2$ or $\mathbb{R}^3 \to \mathbb{R}^3$; it should be clear from the context which is which when it actually matters.

(a) $T(\boldsymbol{v}) = (v_2, v_1)$	(b) $T(\boldsymbol{v}) = (v_1, v_1)$	(c) $T(\boldsymbol{v}) = (0, v_1)$
(d) $T(\boldsymbol{v}) = (0, 1)$	(e) $T(\boldsymbol{v}) = v_1 - v_2$	(f) $T(\boldsymbol{v}) = v_1 v_2$
(g) $T(\boldsymbol{v}) = \boldsymbol{v}/||\boldsymbol{v}||$	(h) $T(\boldsymbol{v}) = v_1 + v_2 + v_3$	(i) $T(\boldsymbol{v}) = (v_1, 2v_2, 3v_3)$
(j) $T(\boldsymbol{v}) = \max\{v_i\}$		

   For those above that are indeed linear, write down the matrix $\boldsymbol{A}$ of this transformation with respect to the standard unit basis vectors.

3. Let V be the space of all polynomials of degree at most 3. Let $T\colon V \to V$ be the derivative operator $T = d/dx$ and $S\colon V \to V$ the second derivative operator $S = d^2/dx^2$. Use $1, x, x^2, x^3$ as both the input basis $\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ and the output basis $\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$.
   (a) Write $T\boldsymbol{v}_0$, $T\boldsymbol{v}_1$, $T\boldsymbol{v}_2$, $T\boldsymbol{v}_3$ in terms of the $\boldsymbol{w}$'s and find the 4×4 matrix $\boldsymbol{A}$ for T.
   (b) Write $S\boldsymbol{v}_0$, $S\boldsymbol{v}_1$, $S\boldsymbol{v}_2$, $S\boldsymbol{v}_3$ in terms of the $\boldsymbol{w}$'s and find the 4×4 matrix $\boldsymbol{B}$ for T.
   (c) Compute the matrices $\boldsymbol{A}^2$, $\boldsymbol{AB}$, $\boldsymbol{BA}$, and $\boldsymbol{B}^2$. Which linear transformation (differential operator) do each of these products represent?

4. Let $T\colon V \to W$ be a linear transformation with $\dim V = \dim W = 3$.
   (a) Using input basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ and output basis $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$, suppose $T(\boldsymbol{v}_1) = \boldsymbol{w}_2$ and $T(\boldsymbol{v}_2) = T(\boldsymbol{v}_3) = \boldsymbol{w}_1 + \boldsymbol{w}_3$. Find the matrix $\boldsymbol{A}$ for this transformation and multiply $\boldsymbol{A}$ by $(1, 1, 1)$. What is $T(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3)$?
   (b) The *kernel* of T, denoted $\ker T$, is defined as the set of all (input) vectors $\boldsymbol{v}$ for which $T(\boldsymbol{v}) = \boldsymbol{0}$. Find $\ker T$ and the nullspace of $\boldsymbol{A}$. Find all solutions to $T(\boldsymbol{v}) = \boldsymbol{w}_2$.
   (c) Find a vector (c_1, c_2, c_3) that is not in the column space of $\boldsymbol{A}$. Find a combination of the $\boldsymbol{w}$'s that is not in the range of T.

5. Give an explicit example of a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with input basis $\boldsymbol{v}_1, \boldsymbol{v}_2$ and output basis $\boldsymbol{w}_1, \boldsymbol{w}_2$ such that the matrix for T is $\boldsymbol{A}$ but the matrix for T^2 is *not* $\boldsymbol{A}^2$. [*Hint*: This will never work unless the $\boldsymbol{v}$'s are different from the $\boldsymbol{w}$'s.]

6. (a) What matrix $\boldsymbol{M}$ transforms $(1,0)$ into (r,t) and $(0,1)$ into (s,u)?
   (b) What matrix $\boldsymbol{N}$ transforms (a,c) into $(1,0)$ and (b,d) into $(0,1)$?
   (c) What condition on a,b,c,d will make Part (b) impossible?
   (d) What matrix (in terms of $\boldsymbol{M}$ and $\boldsymbol{N}$) transforms (a,c) into (r,t) and (b,d) into (s,u)?
   (e) What matrix transforms $(2,5)$ into $(1,1)$ and $(1,3)$ into $(0,2)$? Draw the "grid picture" of this transformation.

7. Let $T\colon V \to W$ be a linear transformation with $\dim V = n$ and $\dim W = m$. Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ be any basis for V. Describe precisely how to pick a basis $\boldsymbol{w}_1, \dots, \boldsymbol{w}_m$ for W so that the $m \times n$ matrix of T in block form is

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{I}_k & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},$$

where $\boldsymbol{I}_k$ is the $k \times k$ identity matrix. The other three entries are blocks of zeros which are potentially empty (depending on the sizes k, n, and m). What does the number k represent?

8. Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ with matrix representation $\boldsymbol{M} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ -3 & 0 & 3 \end{bmatrix}$ with respect to the standard basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$.
   (a) What is the matrix representation $\boldsymbol{A}$ of T with respect to the input basis $\boldsymbol{v}_1 = (1,-1,0)$, $\boldsymbol{v}_2 = (0,1,-1)$, $\boldsymbol{v}_3 = (1,0,1)$? and standard output basis $\boldsymbol{w}_1 = \boldsymbol{e}_1$, $\boldsymbol{w}_2 = \boldsymbol{e}_2$, $w_3 = \boldsymbol{e}_3$?
   (b) What is the matrix representation $\boldsymbol{B}$ of T with respect to the standard input basis $\boldsymbol{v}_1 = \boldsymbol{e}_1$, $\boldsymbol{v}_2 = \boldsymbol{e}_2$, $\boldsymbol{v}_3 = \boldsymbol{e}_3$ and the output basis $\boldsymbol{w}_1 = (1,-1,0)$, $\boldsymbol{w}_2 = (0,1,-1)$, $\boldsymbol{w}_3 = (1,0,1)$?
   (c) What is the matrix representation $\boldsymbol{C}$ of T with respect to the basis $\boldsymbol{v}_1 = \boldsymbol{w}_1 = (1,-1,0)$, $\boldsymbol{v}_2 = \boldsymbol{w}_2 = (0,1,-1)$, $\boldsymbol{v}_3 = \boldsymbol{w}_3 = (1,0,1)$?
   (d) For each of the three parts above, sketch a *commutative diagram* relating the matrices $\boldsymbol{M}$ with (not necessarily all of) $\boldsymbol{I}$, $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ and the matrix $\boldsymbol{S}$ whose columns are $(1,-1,0)$, $(0,1,-1)$, and $(1,0,1)$.