Read: Strang, Section 6.7, 7.1, 7.2, 7.3.


1. Consider the following matrices: \( A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}, \quad A A^T = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}. \)

   (a) Find the eigenvalues \( \sigma_1^2, \sigma_2^2 \) and unit eigenvectors \( v_1, v_2 \) of \( A^T A \).

   (b) For the \( \sigma_i \neq 0 \), compute \( u_i = A v_i / \sigma_i \) and verify that indeed \( || u_i || = 1 \). Find the other \( u_i \) by computing the other unit eigenvector of \( A A^T \).

   (c) Write out the singular value decomposition (SVD), \( A = U \Sigma V^T \).

   (d) Write down orthonormal bases for the four fundamental subspaces of \( A \).

   (e) Describe all matrices that have the same four fundamental subspaces.

2. Which of these transformations satisfy \( T(v + w) = T(v) + T(w) \) and which satisfy \( T(cv) = cT(v) \)? Assume that the mapping is either \( \mathbb{R}^2 \to \mathbb{R}^2 \) or \( \mathbb{R}^3 \to \mathbb{R}^3 \); it should be clear from the context which is which when it actually matters.

   (a) \( T(v) = (v_2, v_1) \)

   (b) \( T(v) = (v_1, v_1) \)

   (c) \( T(v) = (0, v_1) \)

   (d) \( T(v) = (0, 1) \)

   (e) \( T(v) = v_1 - v_2 \)

   (f) \( T(v) = v_1 v_2 \)

   (g) \( T(v) = v / ||v|| \)

   (h) \( T(v) = v_1 + v_2 + v_3 \)

   (i) \( T(v) = (v_1, 2v_2, 3v_3) \)

   (j) \( T(v) = \max \{ v_i \} \)

   For those above that are indeed linear, write down the matrix \( A \) of this transformation with respect to the standard unit basis vectors.

3. Let \( V \) be the space of all polynomials of degree at most 3. Let \( T: V \to V \) be the derivative operator \( T = d/dx \) and \( S: V \to V \) the second derivative operator \( S = d^2/dx^2 \). Use \( 1, x, x^2, x^3 \) as both the input basis \( v_0, v_1, v_2, v_3 \) and the output basis \( w_0, w_1, w_2, w_3 \).

   (a) Write \( T v_0, T v_1, T v_2, T v_3 \) in terms of the \( w \)'s and find the \( 4 \times 4 \) matrix \( A \) for \( T \).

   (b) Write \( S v_0, S v_1, S v_2, S v_3 \) in terms of the \( w \)'s and find the \( 4 \times 4 \) matrix \( B \) for \( T \).

   (c) Compute the matrices \( A^2, AB, BA, \) and \( B^2 \). Which linear transformation (differential operator) do each of these products represent?

4. Let \( T: V \to W \) be a linear transformation with \( \dim V = \dim W = 3 \).

   (a) Using input basis \( v_1, v_2, v_3 \) and output basis \( w_1, w_2, w_3 \), suppose \( T(v_1) = w_2 \) and \( T(v_2) = T(v_3) = w_1 + w_3 \). Find the matrix \( A \) for this transformation and multiply \( A \) by \( (1, 1, 1) \). What is \( T(v_1 + v_2 + v_3) \)?

   (b) The kernel of \( T \), denoted \( \ker T \), is defined as the set of all (input) vectors \( v \) for which \( T(v) = 0 \). Find \( \ker T \) and the nullspace of \( A \). Find all solutions to \( T(v) = w_2 \).

   (c) Find a vector \((c_1, c_2, c_3)\) that is not in the column space of \( A \). Find a combination of the \( w \)'s that is not in the range of \( T \).
5. Give an explicit example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with input basis $v_1, v_2$ and output basis $w_1, w_2$ such that the matrix for $T$ is $A$ but the matrix for $T^2$ is not $A^2$.  

[Hint: This will never work unless the $v$’s are different from the $w$’s.]

6. (a) What matrix $M$ transforms $(1, 0)$ into $(r, t)$ and $(0, 1)$ into $(s, u)$?

(b) What matrix $N$ transforms $(a, c)$ into $(1, 0)$ and $(b, d)$ into $(0, 1)$?

(c) What condition on $a, b, c, d$ will make Part (b) impossible?

(d) What matrix (in terms of $M$ and $N$) transforms $(a, c)$ into $(r, t)$ and $(b, d)$ into $(s, u)$?

(e) What matrix transforms $(2, 5)$ into $(1, 1)$ and $(1, 3)$ into $(0, 2)$? Draw the “grid picture” of this transformation.

7. Let $T : V \to W$ be a linear transformation with dim $V = n$ and dim $W = m$. Let $v_1, \ldots, v_n$ be any basis for $V$. Describe precisely how to pick a basis $w_1, \ldots, w_m$ for $W$ so that the $m \times n$ matrix of $T$ in block form is

$$M = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix},$$

where $I_k$ is the $k \times k$ identity matrix. The other three entries are blocks of zeros which are potentially empty (depending on the sizes $k$, $n$, and $m$). What does the number $k$ represent?

8. Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ with matrix representation $M = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ -3 & 0 & 3 \end{bmatrix}$ with respect to the standard basis $e_1, e_2, e_3$.

(a) What is the matrix representation $A$ of $T$ with respect to the input basis $v_1 = (1, -1, 0), v_2 = (0, 1, -1), v_3 = (1, 0, 1)$? and standard output basis $w_1 = e_1, w_2 = e_2, w_3 = e_3$?

(b) What is the matrix representation $B$ of $T$ with respect to the standard input basis $v_1 = e_1, v_2 = e_2, v_3 = e_3$ and the output basis $w_1 = (1, -1, 0), w_2 = (0, 1, -1), w_3 = (1, 0, 1)$?

(c) What is the matrix representation $C$ of $T$ with respect to the basis $v_1 = w_1 = (1, -1, 0), v_2 = w_2 = (0, 1, -1), v_3 = w_3 = (1, 0, 1)$?

(d) For each of the three parts above, sketch a commutative diagram relating the matrices $M$ with (not necessarily all of) $I, A, B$, and $C$ and the matrix $S$ whose columns are $(1, -1, 0), (0, 1, -1)$, and $(1, 0, 1)$.