2 Key ideas of linear algebra   Read Section 1.3

Objects: Vectors, linear maps (matrices), subspaces.

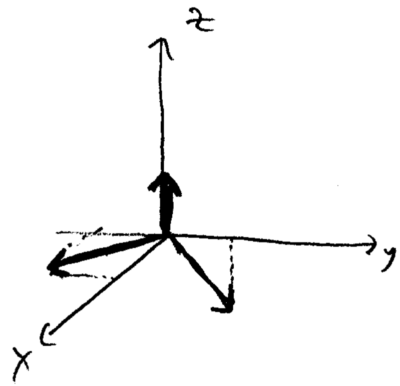
* Vectors We can add & subtract vectors, and multiply them by scalars.

Together, these operations create linear combinations.

Example 1: Given $\underbrace{u, v, w}_{\text{vectors}}$, consider $x_1 u + x_2 v + x_3 w$

scalars

Say $u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

For example, $A \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

Easy: Given $\bar{x}$, get $\bar{b}$.

Harder: Given $\bar{b}$, solve for $\vec{x}$.

Let's solve for $\vec{x}$:

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_3 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

<u>Solution</u>

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix}}$$

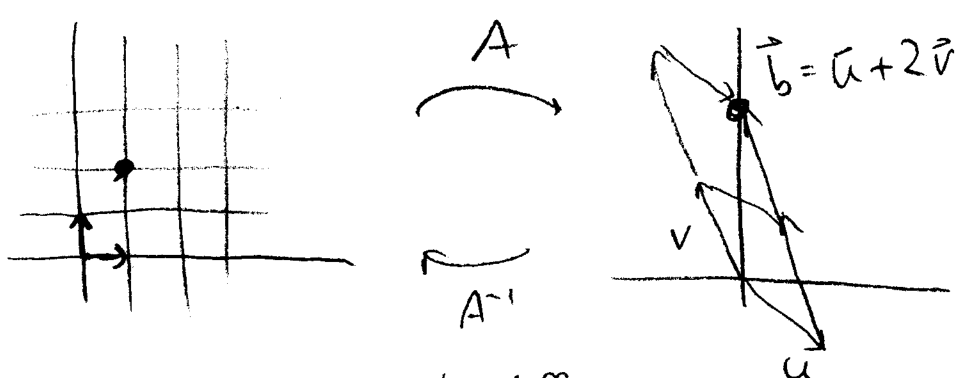
Note that the <u>unique</u> sol'n can be written as: $\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

What we're (secretly doing).

To solve $A\vec{x} = \vec{b}$

we get $\vec{x} = A^{-1}\vec{b}$ (provided that A^{-1} exists).

<u>Geometrically</u>:



<u>Example 2</u>: Let $C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ ← only difference

Say $w^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

There can <u>only</u> be a sol'n to $C\vec{x} = \vec{b}$ if $\vec{b}$ satisfies this:

$$\Downarrow$$

$$0 = b_1 + b_2 + b_3$$

Note: In our first example, $A\vec{x}=\vec{0}$ had one sol'n: $\vec{x}=\vec{0}$.

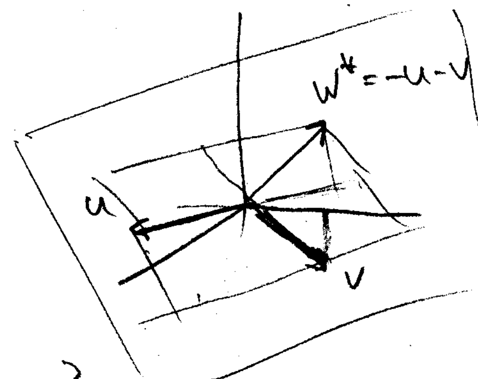
Here, we have many solutions to $C\vec{x}=\vec{0}$.

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} c \\ c \\ c \end{bmatrix}}_{\text{For any } c.} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The set of solutions is the line $\left\{ c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : c \in \mathbb{R} \right\}$.

Claim: This happens because $\{u, v, w^*\}$ only spans a plane.

Reason: $$1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



$$w^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1u - 1v$$

so w^* lies on the plane spanned by $\{u, v\}$. (The plane $b_1+b_2+b_3=0$ or $x+y+z=0$)

We say that $\{u, v, w^*\}$ are linearly dependent.

In some sense "redundant."

Compare to Example 1, where $\{u, v, w\}$ were linearly independent.

We say that $\{u, v, w\}$ is a basis for $\mathbb{R}^3$, because it

- spans (generates all of) $\mathbb{R}^3$
- is linearly independent (non redundant)

Equivalently, if we put u, v, w into a matrix, it will be <u>invertible</u> (we can "undo" that grid picture.)

<u>Think</u>: How does this generalize to $\mathbb{R}^n$?

<u>Key concept</u>: $\boxed{\text{Vector space}}$ – A collection of vectors "closed" under taking linear combinations.

A <u>subspace</u> is a vector space "contained in a bigger one."

<u>Example</u>: What are the subspaces of $\mathbb{R}^3$?

<u>Ans</u>:

3D: $\mathbb{R}^3$

2D: planes through $\vec{0}$

1D: lines through $\vec{0}$

0D: The point $\vec{0}$.