(4) Multiplication & Inverse matrices

Multiplication of matrices (4 ways):

① Rows times columns:

$C_{34} = (\text{Row } 3)\cdot(\text{Col. } 4)$

$$\begin{array}{c} \\ \text{row } 3 \\ \\ \end{array}\begin{bmatrix} & & \\ \text{———} & \text{———} & \text{———} \\ & & \end{bmatrix}_{m\times n}\begin{array}{c}\text{Col. } 4 \\ \begin{bmatrix} & \| & \\ & \| & \\ & \| & \end{bmatrix}_{n\times p}\end{array} = \begin{bmatrix} & & & \\ & & & \bullet \\ & & & \end{bmatrix}_{m\times p}$$

$$C_{34} = (\text{Row } 3 \text{ of } A)\cdot(\text{Column } 4 \text{ of } B)$$
$$= a_{31}b_{14} + a_{32}b_{24} + \dots + a_{3n}b_{n4} = \sum_{k=1}^{n} a_{3k}b_{k4}.$$

② By columns:

$$\begin{bmatrix} \\ A \\ \\ \end{bmatrix}\overset{B}{\begin{bmatrix} \| & \cdots & \| \end{bmatrix}} = \underbrace{\begin{bmatrix} A\vec{b}_1 & \cdots & A\vec{b}_p \end{bmatrix}}_{\text{These columns of } C \text{ are linear combinations of columns of } A.}$$

These columns of C are linear combinations of columns of A.

③ By rows:

$$\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_m^T \end{bmatrix}\begin{bmatrix} \\ B \\ \\ \end{bmatrix} = \begin{bmatrix} \vec{a}_1^T B \\ \vdots \\ \vec{a}_m^T B \end{bmatrix}$$

← each row is a linear combin. of the rows of B.

$m \times 1$ $\quad$ $1 \times p$

④ Sum of (Columns of A) × (Rows of B).

Ex: $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}_{3\times 1} \begin{bmatrix} 1 & 6 \end{bmatrix}_{1\times 2} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}_{3\times 2}$

$$\underset{A}{\begin{bmatrix} \Big\| & \cdots & \Big\| \end{bmatrix}} \underset{B}{\begin{bmatrix} = \\ \vdots \\ = \end{bmatrix}} = \sum_{j,k=1}^{n} \underbrace{\vec{a}_j^{\,T} \vec{b}_k}_{m\times p \text{ Matrix}}$$

n columns $\qquad$ n rows

sum of n^2 terms.

<u>Inverses</u>: (square matrices only).

The inverse of a matrix A, denoted A^{-1} satisfies

$$A^{-1}A = I = AA^{-1}.$$

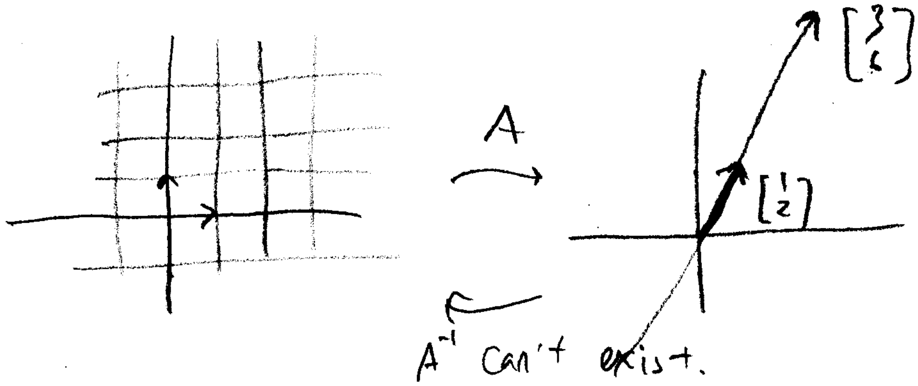
If A^{-1} exists, A is said to be <u>invertible</u> or <u>nonsingular</u>.

<u>Singular case</u>:

<u>Ex</u>: $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ has no inverse.

<u>Why</u>? ① You can't get $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ from a linear combination of the columns, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

(2) "Grid" picture:



(3) There is a vector $\vec{x} \neq \vec{0}$ with $A\vec{x} = \vec{0}$: $\begin{bmatrix}1 & 3\\2 & 6\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

* Anytime this happens, A^{-1} can't exist!

<u>Why</u>: Suppose A^{-1} existed, and $A\vec{x} = \vec{0}$, with $\vec{x} \neq \vec{0}$.

Then $\vec{x} = I\vec{x} = A^{-1}A\vec{x} = A\vec{0} = \vec{0}$. (Contradiction). ✓

Now, consider a matrix that has an inverse

$$\begin{bmatrix}1 & 3\\2 & 7\end{bmatrix}\begin{bmatrix}a & c\\b & d\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}.$$

$$A \quad A^{-1} = I$$

Get 2 eqns (doing this by columns):

$$\begin{bmatrix}1 & 3\\2 & 7\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}1 & 3\\2 & 7\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

Make an "augmented" matrix:

$$\underset{[A \,\vdots\, I]}{\left[\begin{array}{cc:cc}1 & 3 & 1 & 0\\2 & 7 & 0 & 1\end{array}\right]} \xrightarrow{E_{21}} \left[\begin{array}{cc:cc}1 & 3 & 1 & 0\\0 & 1 & -2 & 1\end{array}\right] \xrightarrow{E_{12}} \underset{[I \,\vdots\, A^{-1}]}{\left[\begin{array}{cc:cc}1 & 0 & 7 & -3\\0 & 1 & -2 & 1\end{array}\right]}$$

(<u>why</u>?)