9 Linear independence, spanning sets, & bases

Suppose A is an $m\times n$ matrix, $m<n$: $\begin{bmatrix} & A & \end{bmatrix}_{m\times n}$

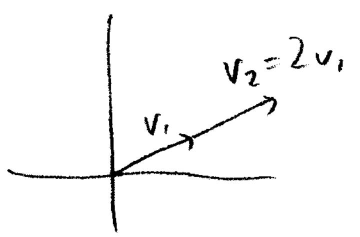
Then there are nonzero sol'ns to $Ax=0$.

Reason:
- Algebraically: There will be free vars (can be anything!)
- Geometrically: At least one dimension must collapse.

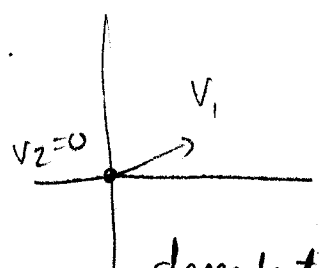
Def: Vectors $v_1,\dots,v_n$ are (linearly) independent if no non-trivial combination gives the zero vector.

i.e., $c_1v_1+c_2v_2+\dots+c_nv_n=0 \implies c_1=c_2=\dots=c_n=0$.

Ex.

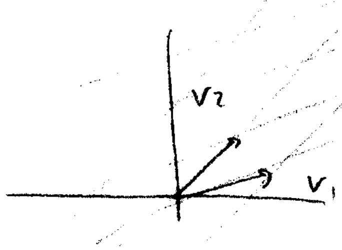


dependent: $2v_1-v_2=0$

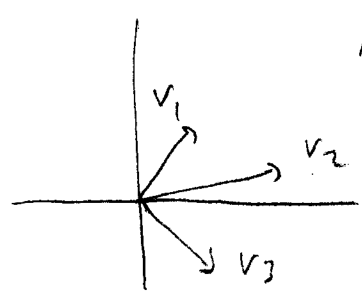


anything

dependent: $0v_1+3v_2=0$



independent.



dependent

$$Ac=\begin{bmatrix}1&3&2\\2&1&-2\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\implies c_1v_1+c_2v_2+c_3v_3=0.$$

(2)

<u>Remark</u>: If a set of vectors is <u>dependent</u>, then we can write one in terms of the others.

<u>Ex</u>: $$1\begin{bmatrix}1\\2\\3\end{bmatrix} - 2\begin{bmatrix}4\\5\\6\end{bmatrix} + 1\begin{bmatrix}7\\8\\9\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}7\\8\\9\end{bmatrix} = \underbrace{-1\begin{bmatrix}1\\2\\3\end{bmatrix} + 2\begin{bmatrix}4\\5\\6\end{bmatrix}}$$

on the "grid" of these two.

Yet another characterization:

* The vectors $v_1, \ldots, v_n$ are <u>independent</u> if the nullspace of the matrix $\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ is $\{\vec{0}\}$.

They are <u>dependent</u> if $Ac = 0$ for some $c \neq 0$.

<u>Note</u>: If cols. are independent, then rank $A = n$ (full rank)

If cols. are dependent, then rank $A < n$.

<u>Def</u>: Vectors $v_1, \ldots, v_k$ <u>span</u> a space if the set of all (linear) combinations of those vectors = the space.

[<u>Think</u>: $v_1, \ldots, v_k$ spans (or "generates") the "grid" they determine.

Def: A <u>basis</u> of a vector space is a set $v_1, \dots, v_d$ of vectors with 2 properties:

1. They are <u>independent</u> ("not too many")
2. They <u>span</u> the space ("not too few")

<u>Example</u>: Space is $V = \mathbb{R}^3$.

One basis is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Another basis: $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix}$. ← anything not in the plane spanned by v_1 & v_2

<u>Q</u>: How to check that this is a basis?

<u>A</u>: Row-reduce the matrix: $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 5 \\ 3 & 1 & 5 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ yes ✓ $\qquad$ $\begin{bmatrix} & — & \\ & & \\ 0 & 0 & 0 \end{bmatrix}$ no ✗

<u>Q</u>: Is $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ a basis for some vector space?

<u>A</u>: <u>Yes</u>. For the plane <u>spanned</u> by these vectors.

<u>Fact</u>: Every basis of a space has the same # of vectors.
Call this number the <u>dimension</u> of the space.

<u>Ex</u>: Let $V = C(A)$, where $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$.

Do the column vectors span $C(A)$? [Ans: <u>Yes</u>]

Are they a basis? [Ans: <u>No</u>]

What's a basis? Ans: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Basis for $N(A)$: $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

<u>Theorem</u>: $\text{rank}(A)$ = dimension of $C(A)$.

(# of pivot cols)

<u>Recall</u>: The "special solutions" are a basis for $N(A)$.

Thus,

$$\boxed{\begin{aligned} \dim N(A) &= \#\text{ free vars} = n - r \\ \dim C(A) &= \#\text{ pivot vars} = r \end{aligned}}$$