$\boxed{10}$ Four fundamental subspaces

Let A be an $m \times n$ matrix: $\underbrace{\begin{bmatrix} & A & \end{bmatrix}}_{n \text{ columns}} \Big\}\, m \text{ rows.}$

Column space $\boxed{C(A)}$ in $\mathbb{R}^m$

Null space $\boxed{N(A)}$ in $\mathbb{R}^n$

Row space = all combins. of rows of A

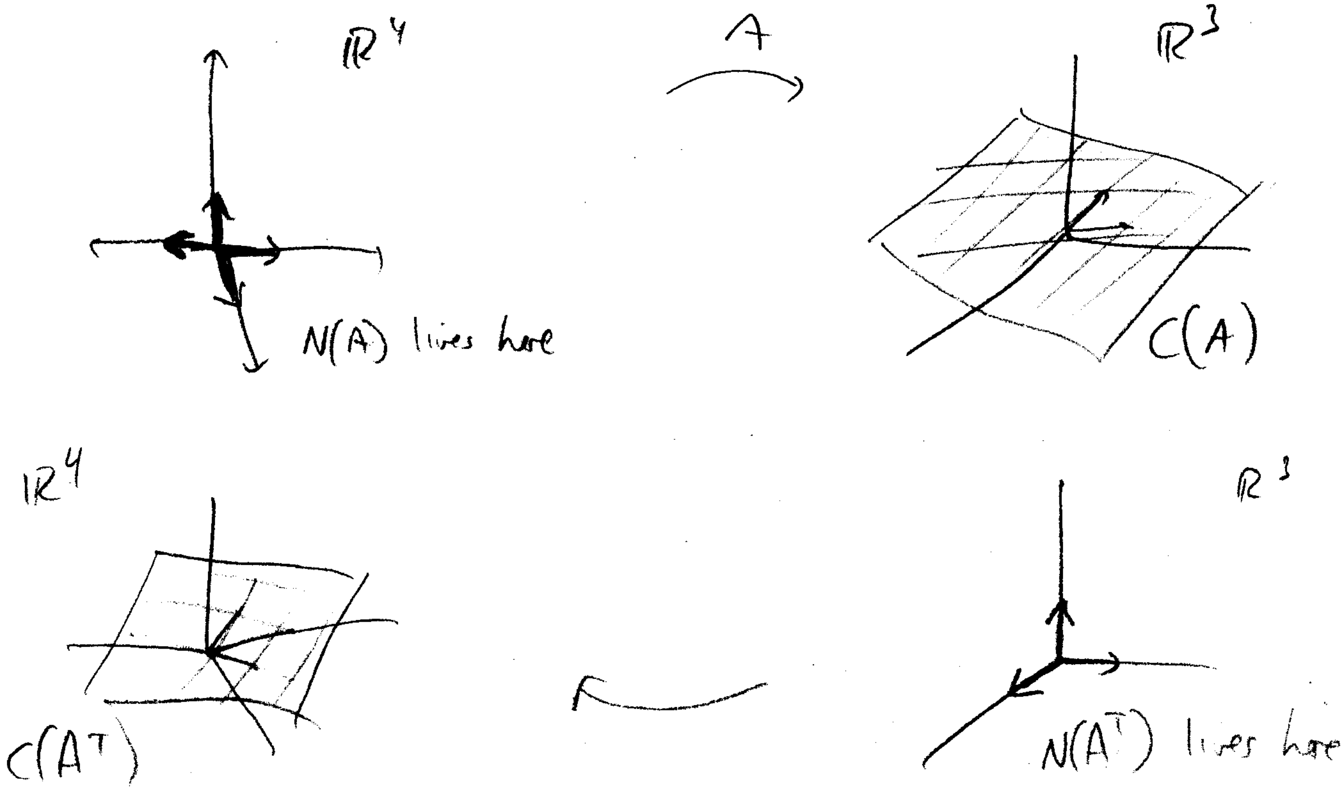
= all combins. of cols. of A^T = $\boxed{C(A^T)}$ in $\mathbb{R}^m$

Null space of A^T = $\boxed{N(A^T)}$ in $\mathbb{R}^n$

("left nullspace").

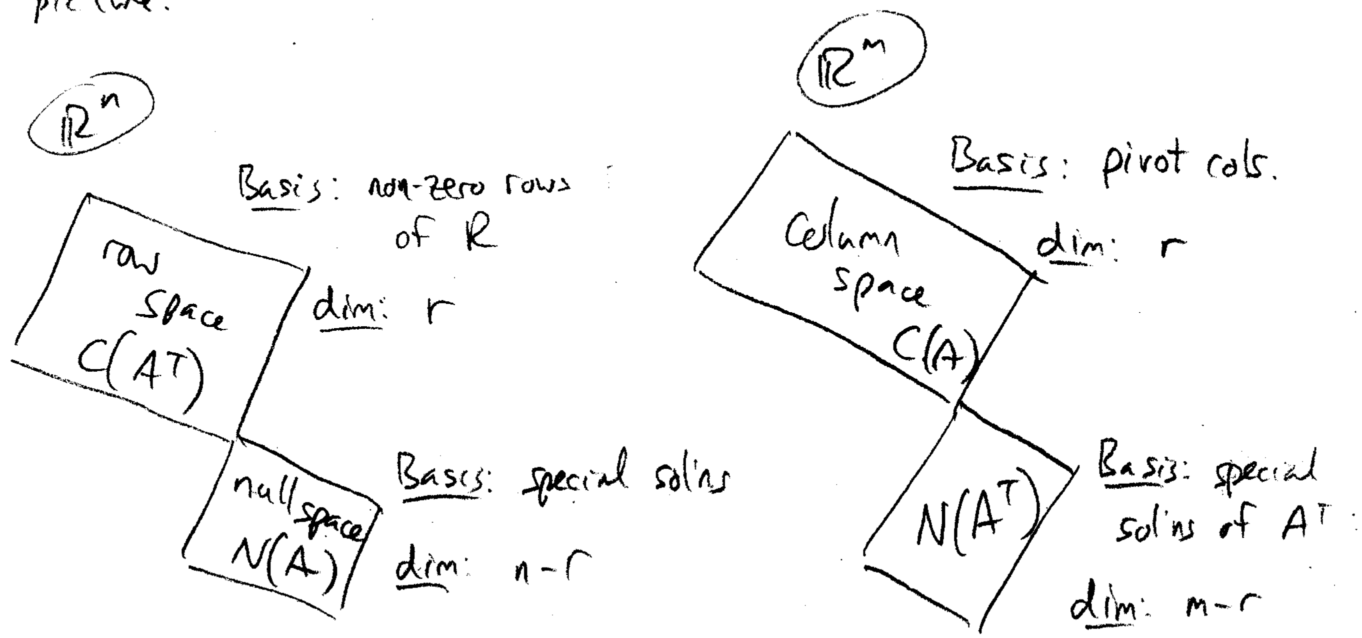
Grid picture:

Ex:



Goal: Understand these spaces (e.g., bases, dimension)

Another picture:



Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \quad I \qquad\quad F \\ \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \boxed{\begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} \end{bmatrix} \end{array} = R$$

Note: $C(A) \neq C(R)$, but the row spaces are the same!

Basis for rowspace is first r rows of R (r = rank A).

4th space: $N(A^T)$; "left nullspace"

$$A^T y = 0 \qquad \begin{bmatrix} A^T \end{bmatrix}\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\Rightarrow y^T A = 0^T \qquad \begin{bmatrix} y^T \end{bmatrix}\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 0^T \end{bmatrix}$$

Recall: If A is a square invertible matrix, then we can find A^{-1} by row reducing $[A \ I] \longrightarrow [I \ A^{-1}]$.

Now, suppose A is $m \times n$. Consider the augmen

Consider the "augmented matrix" $[A_{m\times n} \ I_{m\times m}]$.

Elimination puts it in reduced-row echelon form:

$$[A_{m\times n} \ I_{m\times m}] \xrightarrow{E} [R_{m\times n} \ E_{m\times m}] \quad (\text{so, } EA = R).$$

Ex: $$\left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

Check: $$\begin{bmatrix} -1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

in left-nullspace

Note: $\dim N(A^T) = m - r = 3 - 2 = 1$

Basis: $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$