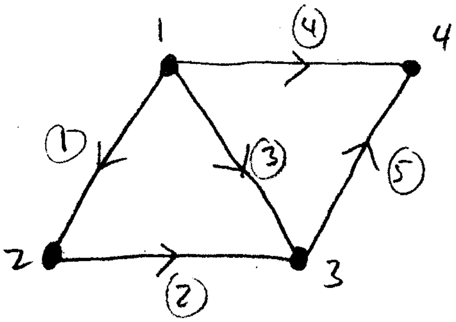
12 Graphs and networks

Graph: Nodes & edges.

Ex:



Incidence matrix A:

cycle

Cycle

$$
\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}
\begin{array}{l} \text{edge } 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
$$

node 1, 2, 3, 4

Nullspace:

$\boxed{Ax = 0}$

$$
\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
=
\begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix}
\qquad
x_n = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

potentials at the nodes

potential differences along each edge.

So, rank $A = 3$.

Left nullspace: $\boxed{A^T y = 0}$ "Most fundamental eq'n in applied math."

$\boxed{2}$

$\dim N(A^T) = m - r = 5 - 3 = \boxed{2}$.

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boxed{A^T y = 0}$$

$=0$ if no current sources

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ potential at nodes $\xrightarrow{A^T C A x = F}$ $A^T y = F$ "Kirchoff's current law" (KCL)

$\downarrow A$ $\qquad\qquad\qquad\qquad\qquad \uparrow A^T$

$x_2 - x_1$, etc. potential diff. $\xrightarrow[\text{Ohm's law, } I = V/R.]{C}$ currents $y_1, \ldots, y_m$ on edges.

(C is a diagonal matrix of edge conductances).

$A^T y = 0$:
$$\begin{cases} -y_1 - y_3 - y_4 = 0 \\ y_1 - y_2 = 0 \\ y_2 + y_3 - y_5 = 0 \\ y_4 - y_5 = 0 \end{cases}$$

Note: Elimination will yield a zero-row. (why?)

Basis for $N(A^T)$:

$$\text{loop 1} \longrightarrow \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \longleftarrow \text{loop.}$$

Note that the "big loop" is the sum of these.

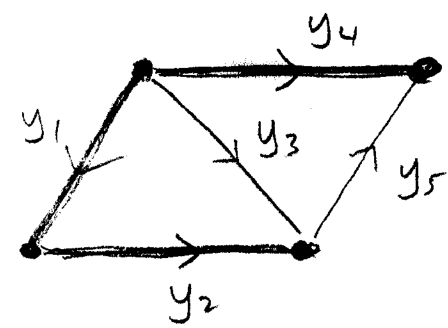
<u>Summary</u>: $$(A^T \overbrace{C\, \underbrace{A)\, x}_{\text{potential diff.}}}^{y} = f$$

<u>Notes</u>: • A^TCA is symmetric.

Flow from "high-to-low" potential

- The entries of C are c_{ii} = conductances along edge i
- Physically current is actually $\boxed{y = -CAx}$ (<u>not</u> CAx)
- In Mechanical engineering, the c_{ii}'s are elasiticities; this is "Hooke's law"!)

<u>Row space</u>: Basis: Col 1, 2, 4



The corresponding edges are a <u>tree</u> (graph with no loops).

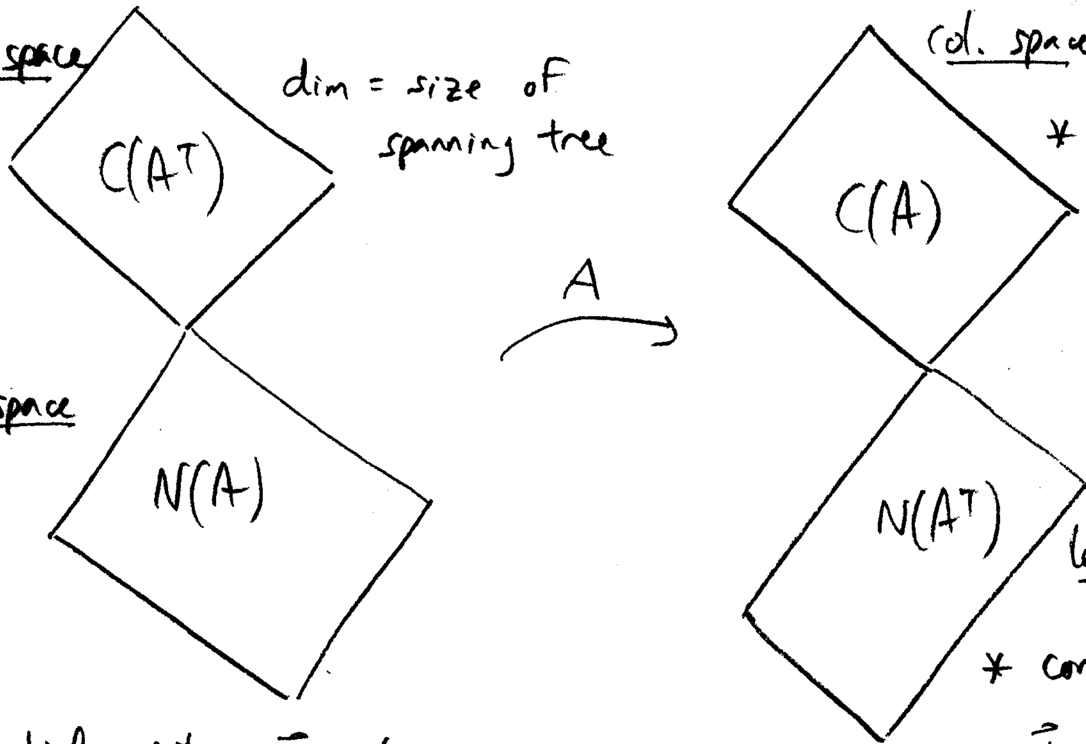
$\dim N(A^T) = m - r$.

$\#\text{ loops} = \#\text{ edges} - (\#\text{ nodes} - 1)$

$\Rightarrow$ $\boxed{(\#\text{ nodes}) - (\#\text{ edges}) + (\#\text{ loops}) = 1}$ "Euler's formula

Works for any graph!

Summary



row space: dim = size of spanning tree

null space: potential vectors $\vec{x}$ s.t. each potential diff is zero.

dim = # connected components

col. space: * contains $\vec{b}$ iff $\vec{b}$ satisfies KVL

left null space: * contains $\vec{y}$ iff $\vec{y}$ satisfies KCL

dim = # loops