Diagonalization

Big idea: \( A = S \Lambda S^{-1} \)

Suppose we have a linearly independent set of eigenvectors of \( A \).

Put them into the cols. of \( S \), the eigenvector matrix.

Note: \( AS = A [v_1 \ v_2 \ldots \ v_n] = [\lambda_1 v_1 \ \lambda_2 v_2 \ldots \ \lambda_n v_n] \)

\[ = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} = S \Lambda. \]

Thus, \( AS = S \Lambda \Rightarrow A = S \Lambda S^{-1} \quad \text{or} \quad S^T A S = \Lambda. \)

Grid picture:

\[ A^2 = S \Lambda S^{-1} S \Lambda S^{-1} = S \Lambda^2 S^{-1} \quad \text{or} \quad A^k = S \Lambda^k S^{-1} \]
Power of matrices: \( Ax = \lambda x \Rightarrow A^2 x = \lambda^2 x. \)

**Summary**: \( A^k \) has:
- Same e-vectors as \( A \)
- Eigenvalues \( \lambda^k \)

**Theorem**: \( A^k \to 0 \) as \( k \to \infty \) iff all \( |\lambda_i| < 1 \).

*(Assumption: we still have \( n \) lin indep. e-vectors.)*

**Main point**: \( A \) is sure to have \( n \) indep. e-vectors (and be diagonalizable) if all \( \lambda_i \)'s are different.

**Repeated eigenvalues**: May or may not have \( n \) indep. e-vectors.

**Problem case**: \( A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \) \( \lambda_1 = \lambda_2 = 2 \)

\[ \text{algebraic multiplicity} = 2 \]

\[ A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{geometric multiplicity} = 1. \]

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*But every vector is an e-vector. So pick \( v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)*
Application: Difference equations.

Start with initial vector $u_0$.

Iteratively multiply $A$: $u_0 \xrightarrow{A} u_1 \xrightarrow{A} u_2 \xrightarrow{A} u_3 \rightarrow \cdots$

$\Rightarrow u_1 = Au_0$, $u_2 = Au_1 = A^2u_0$, in general: $u_{k+1} = A^k u_0$

Better way to find $u_{k+1}$ (for large $k$!)

Write $u_0 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \sum c_i v_i$

$Au_0 = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_n \lambda_n v_n = \sum \lambda_i c_i v_i$

$A^k u_0 = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 + \cdots + c_n \lambda_n^k v_n = \sum \lambda_i^k c_i v_i$.

* We're using $\{v_1, \ldots, v_n\}$ (eigenvectors) as a basis instead of $\{e_1, \ldots, e_n\}$

Ex: Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, ... 

Questions:

- What is $F_{100}$?
- How fast are these growing?

Answers lie in the eigenvalues.

Rule: $F_{k+2} = F_{k+1} + F_k$  

\[ F_{k+2} = F_{k+1} + F_k \]

\[ F_{k+1} + F_{k+1} + F_k \]

\[ \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix} \]

\[ u_{k+1} = A u_k \]
If \( A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \), then 
\[
\det(A - \lambda I) = \lambda^2 - \lambda - 1 = 0
\]

\[
\lambda = \frac{1 \pm \sqrt{5}}{2}
\]

\[
\lambda_1 = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618 \quad \text{(Golden ratio!)}
\]

\[
\lambda_2 = \frac{1}{2}(1 - \sqrt{5}) \approx -0.618
\]

Note: \( A^{100} u_0 = c_1 \lambda_1^{100} v_1 + c_2 \lambda_2^{100} v_2 \Rightarrow F_{100} \approx c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^{100} \approx 0 \)

Given \( A - \lambda I = \begin{bmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \) \( v_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, \) \( v_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \)

\[
U_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 v_1 + c_2 v_2
\]

Solve
\[
\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]