$\boxed{9}$ Diagonalization

Big idea: $A = S \Lambda S^{-1}$

matrix of eigenvectors.

matrix of eigenvalues

Suppose we have n linearly independent e-vectors of A.

Put them into the cols. of S. "eigenvector matrix"

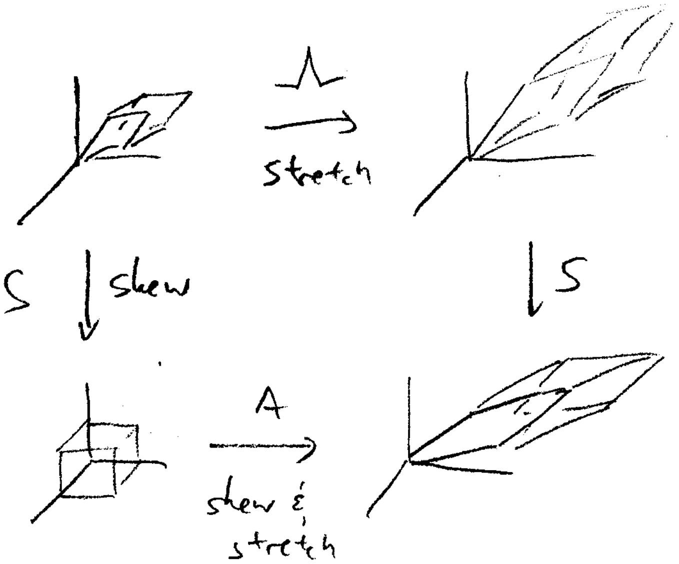
Note: $$AS = A\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix} = S\Lambda.$$

Thus, $AS = S\Lambda \Rightarrow A = S\Lambda S^{-1}$ or $S^{-1}AS = \Lambda$.

Grid picture:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\Lambda"] \arrow[d, "S"'] & \mathbb{R}^n \arrow[r, dashed, "\Lambda"] \arrow[d, "S"] & \mathbb{R}^n \arrow[d, dashed, "S"] \\
\mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \arrow[r, dashed, "A"] & \mathbb{R}^n
\end{tikzcd}$$



$A^2 = S\Lambda S^{-1} S \Lambda S^{-1} = S\Lambda^2 S^{-1}$. $A^k = S\Lambda^k S^{-1}$.

Powers of matrices: $Ax = \lambda x \Rightarrow A^2 x = \lambda A x = \lambda^2 x$.

Summary: A^k has:

- * Same e-vectors as A
- * eigenvalues λ_i^k

Theorem: $A^k \to 0$ as $k \to \infty$ iff all $|\lambda_i| < 1$.

(Assumption: we still have n lin indep. e-vectors.)

Main point: A is sure to have n indep. e-vectors (and be diagonalizable) if all λ's are different.

Repeated eigenvalues: May or may not have n indep. e-vectors.

Problem case: $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $\quad \underbrace{\lambda_1 = \lambda_2 = 2}$

↑ algebraic multiplicity = 2

$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ← geometric multiplicity = 1.

Compare to $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad \lambda_1 = \lambda_2 = 2$.

But every vector is an e-vector. So pick $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Application: Difference equations.

Start with initial vector u_0.

Iteratively multiply A: $u_0 \xrightarrow{A} u_1 \xrightarrow{A} u_2 \xrightarrow{A} u_3 \longrightarrow \cdots$

$\Rightarrow$ $u_1 = Au_0$, $u_2 = Au_1 = A^2 u_0$, in general: $\boxed{u_{k+1} = A^k u_0}$

Better way to find u_{k+1} (for large k!)

$$\text{Write} \quad u_0 = C_1 V_1 + C_2 \,_2 + \dots + C_n \,_n = S\vec{c}$$

$$A u_0 = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \dots + c_n \lambda_n v_n = \Lambda S \vec{c}$$

$$A^k u_0 = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 + \dots + c_n \lambda_n^k v_n = \Lambda^{100} S \vec{c}.$$

* We're using $\{v_1, \dots, v_n\}$ (eigenvectors) as a basis instead of $\{e_1, \dots, e_n\}$.

Ex: Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, \dots$

Questions:
- What is F_{100}?
- How fast are these growing?

Answers lie in the eigenvalues.

$$\text{Rule: } F_{k+2} = F_{k+1} + F_k \quad \overset{\text{trick}}{\rightsquigarrow} \quad \begin{cases} F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_{k+1} + 0 F_k \end{cases}$$

$$\text{let } u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \qquad \underset{u_{k+1} \nearrow}{\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix}} = \underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \underset{\nwarrow u_k}{\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}}$$

[9]

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\det(A-\lambda I) = \lambda^2 - \lambda - \lambda = 0$

$\lambda = \frac{1 \pm \sqrt{5}}{2}$ $\lambda_1 = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$ (Golden ratio!)

$\lambda_2 = \frac{1}{2}(1-\sqrt{5}) \approx -.618$

<u>Note</u>: $A^{100} u_0 = c_1 \lambda_1^{100} v_1 + \underbrace{c_2 \lambda_2^{100} v_2}_{\approx 0} \Rightarrow F_{100} \approx c_1 \left(\frac{1+\sqrt{5}}{2}\right)^{100}$

E-vectors. $A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $v_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

$$u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 v_1 + c_2 v_2}$$

Solve $\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$