11 Application: Fourier Series

Recall: Projections with orthonormal basis: $q_1, \dots, q_n$.

Write any $v = x_1 q_1 + x_2 q_2 + \dots + x_n q_n$.

Multiply through by q_1^T: $\quad q_1^T v = x_1 q_1^T q_1 + 0 + \dots + 0 = x_1$

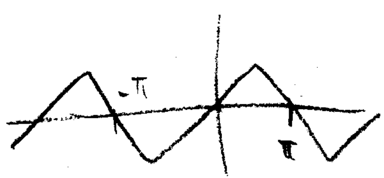
$\Rightarrow x_1 = q_1^T v$, similarly $x_i = q_i^T v$.

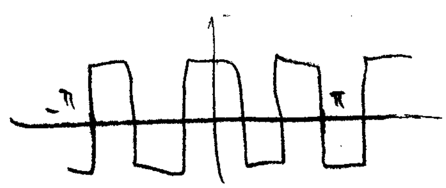
Matrix form:

$$\begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{v}.$$

$$Q \quad x \quad = \quad v \quad \Rightarrow \quad \boxed{x = Q^T v}$$

Fourier series: Consider the set of 2π-periodic functions:





, ...

These form a vector space.

Theorem (Fourier): The functions $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots\}$ form an orthogonal basis of this space!

2

What do we mean by "orthogonal"?

For $\mathbb{R}^n$: $\quad v \cdot w = v^T w = \sum_{i=1}^{n} v_i w_i$

* Functions: $\quad f \cdot g = f^T g = \int_0^{2\pi} f(x)\, g(x)\, dx$

<u>Orthogonality</u>: $\displaystyle \int_0^{2\pi} \cos nx \cos mx \, dx = \begin{cases} 0 & i \neq j \\ \pi & i = j \end{cases}$

$$\int_0^{2\pi} \sin nx \sin mx \, dx = \begin{cases} 0 & i \neq j \\ \pi & i = j \end{cases}$$

$$\int_0^{2\pi} \cos nx \sin mx \, dx = 0$$

For any 2π-periodic function $f(x)$, write

$$f(x) = a_0 1 + a_1 \cos x + b_1 \sin nx + a_2 \cos 2x + b_2 \sin 2x + \ldots$$

To find a_n, multiply by $\cos nx$ & integrate:

$$\int_0^{2\pi} f(x) \cos nx \, dx = \int_0^{2\pi} a_n (\cos nx)^2 dx = a_n \pi$$

$$\Rightarrow a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$$

Similarly, $b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$, $\quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx$.