2 Positive definite matrices

Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Then A is positive definite if one of the following (equivalent) conditions hold.

① $\lambda_1 > 0,\ \lambda_2 > 0$ — eigenvalues

② subdets: $a > 0,\ ac - b^2 > 0$ $\quad \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ — determinants

usually the def'n ③ Pivots: $a > 0,\ \dfrac{ac-b^2}{a} > 0$ — pivots

$\hookrightarrow$ * ④ $x^T A x > 0$ for $x \neq 0$. — quadratic forms

It should be clear what the analogue is for $n \times n$.

Ex: $\begin{bmatrix} 2 & 6 \\ 6 & >18 \end{bmatrix}$ is positive definite (pos. semi-definite if $d=18$)

$\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$ not pos. def'n:

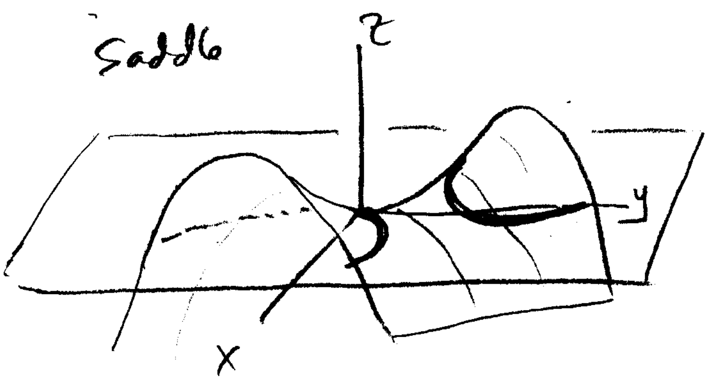
$$[x_1 \quad x_2] \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overset{a}{\overset{\downarrow}{2}} x_1^2 + \overset{2b}{\overset{\downarrow}{12}} x_1 x_2 + \overset{c}{\overset{\downarrow}{7}} x_2^2 \not> 0$$

Take $x_1 = 1,\ x_2 = -1 \rightsquigarrow f(x_1, x_2) = -3$

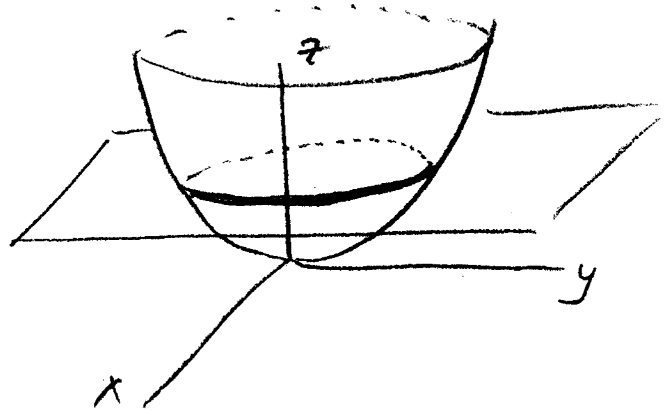
(2)

Graph of $f(x,y) = \vec{x}^T A \vec{x} = ax^2 + 2bxy + cy^2$.

saddle



vs.



cross-section: hyperbola ellipse

Recall calculus

min of $f(x)$: $f'(x) = 0$, $f''(x) > 0$

min of $f(x,y)$: $Df(x,y) = 0$, $H(f) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ is positive definite.

e.g., $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = 2\begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

Ex: $f(x,y) = 2x^2 + 12xy + 20y^2$

$= 2(x+3y)^2 + 2y^2 > 0$ for $(x,y) \neq (0,0)$

come from elimination.

$A = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}}_{U}$

pivot = 2, 2

multiplier = 3

positive pivots

$\Downarrow$

positive squares

$\Downarrow$

positive definite

More properties (think analogues of "positivity" for numbers)

- A pos. def. $\Longrightarrow$ A^{-1} pos. def.

  Why: Eigenvalues of A^{-1} are $1/\lambda$, where $Av = \lambda v$.

- A, B pos. def. $\Longrightarrow$ $A + B$ pos. def.

  Why: $x^T(A+B)x = x^TAx + x^TBx > 0$ if $x \neq 0$.

- M is $m \times n$, rank n (lin. indep. cols.) $\Longrightarrow$ M^TM pos. def.

  Why: $x^T(M^TM)x = x^TM^TMx = (Mx)^T(Mx) = \|Mx\| > 0$ if $x \neq 0$.

Note: If rank $M < n$, then M^TM pos. semi-def.

Application: (Conic sections)

$$\text{let } f(x_1, x_2) = 5x_1 + 8x_1x_2 + 5x_2^2 = \overset{x^T}{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} \overset{A}{\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$\lambda_1 = 9 \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1 \quad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$A = \underset{Q}{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}} \underset{\Lambda}{\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}} \underset{Q^T}{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

45° rotation matrix

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The equation $x^T A x = 1$ is a <u>conic section</u> (ellipse).

$$x^T A x = x^T Q \Lambda Q^T x = 1$$

$$(Q^T x)^T \Lambda (Q^T x) = 1$$

$$y^T \Lambda y = 1 \quad \text{where } y = Q^T x \Rightarrow x = Qy.$$