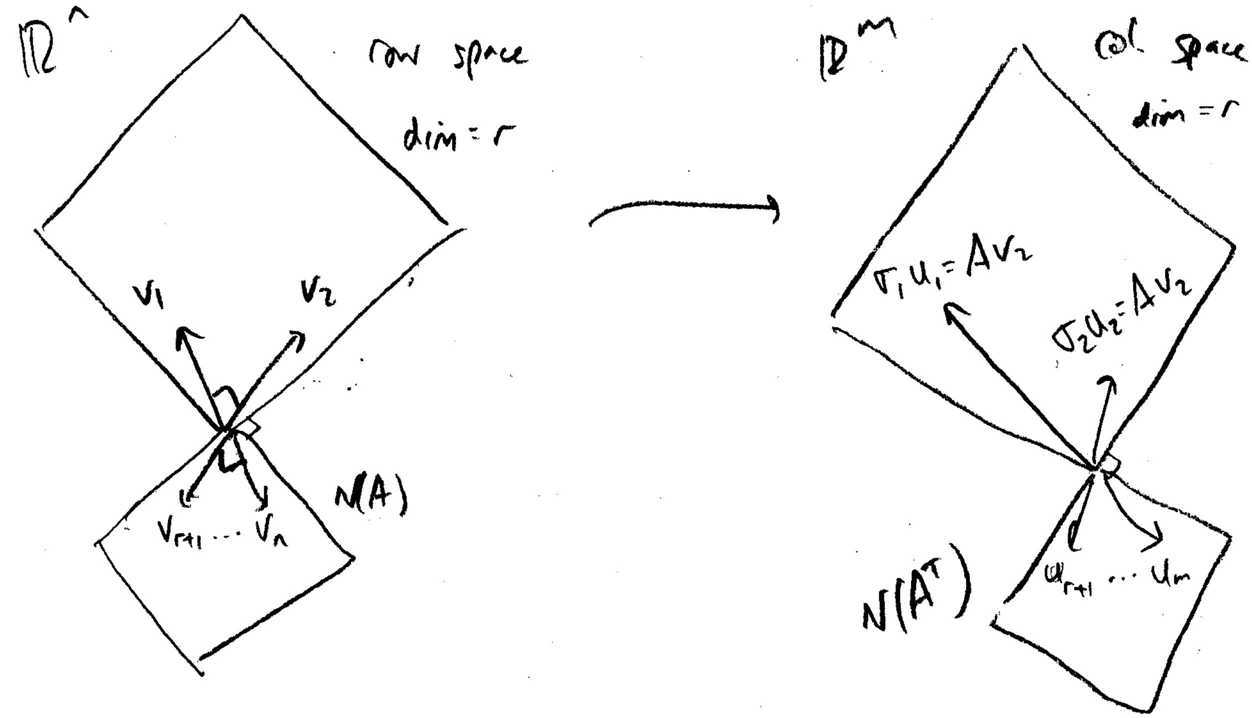
5 Singular value decomposition (SVD)

Big idea: $A = U \Sigma V^T$

Special case: $A = Q \Lambda Q^T$ when A is symmetric.

No good: $A = S \Lambda S^{-1}$ (e-vectors might not be $\perp$)



Goal: Find orthonormal bases of $C(A^T)$ & $C(A)$ so

$$A \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}$$

$$A \quad V \quad = \quad U \quad \Sigma \quad \Rightarrow \boxed{A = U \Sigma V^T}$$

Note: Then we can extend these bases to $N(A)$, $N(A^T)$:

$$A\left[\underbrace{v_1 \cdots v_r}_{\substack{\text{basis of}\\ C(A^T)}} \; \underbrace{v_{r+1} \cdots v_n}_{\substack{\text{basis of}\\ N(A)}}\right] = \left[\underbrace{u_1 \cdots u_r}_{\substack{\text{basis of}\\ C(A)}} \; \underbrace{u_{r+1} \cdots u_m}_{\substack{\text{basis of}\\ N(A^T)}}\right]\begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

Question: How to find U, V, Σ ?

Want $\boxed{Av_i = \sigma_i u_i}$ for $i = 1, \dots, r$

Trick: $$A^TA = (V\Sigma^T U^T)(U\Sigma V^T) = \overbrace{V\begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \end{bmatrix}V^T}^{Q\Lambda Q^T\,!}$$

Symm, pos. (semi)-def.

$$AA^T = (U\Sigma V^T)(V\Sigma U^T) = U\Sigma^2 U^T$$

So, $\boxed{\begin{array}{l} V = \text{eigenvectors of } A^TA \\ U = \text{eigenvectors of } AA^T \end{array}}$ ← find this first.

Ex 1: $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$ $\quad A^TA = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$

$$A^TA\begin{bmatrix}1\\1\end{bmatrix} = 32\begin{bmatrix}1\\1\end{bmatrix}; \quad \sigma_1^2 = 32, \quad v_1 = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

$$A^TA\begin{bmatrix}1\\-1\end{bmatrix} = 18\begin{bmatrix}1\\-1\end{bmatrix}; \quad \sigma_2^2 = 18, \quad v_2 = \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$$

$$AA^T = \begin{bmatrix}4 & 4\\-3 & 3\end{bmatrix}\begin{bmatrix}4 & -3\\4 & 3\end{bmatrix} = \begin{bmatrix}32 & 0\\0 & 18\end{bmatrix}$$

$$Av_1 = \begin{bmatrix}4 & 4\\-3 & 3\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}8/\sqrt{2}\\0\end{bmatrix} = \sqrt{32}\begin{bmatrix}1\\0\end{bmatrix} = \sigma_1 u_1$$

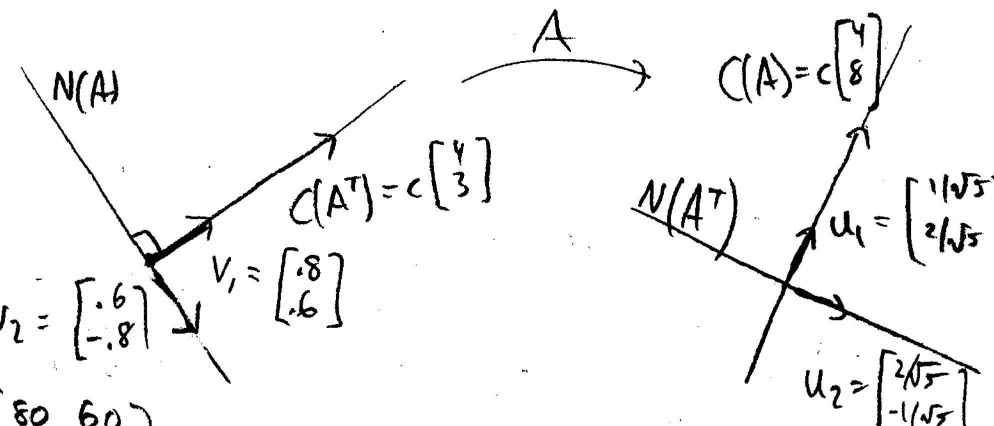
$$Av_2 = \begin{bmatrix}4 & 4\\-3 & 3\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}0\\-6/\sqrt{2}\end{bmatrix} = \sqrt{18}\begin{bmatrix}0\\-1\end{bmatrix} = \sigma_2 u_2$$

So,

$$\underbrace{\begin{bmatrix}4 & 4\\-3 & 3\end{bmatrix}}_{A} = \underbrace{\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}\sqrt{32} & 0\\0 & \sqrt{18}\end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}}_{V^T}$$

Ex 2. (rank 1)

$$A = \begin{bmatrix}4 & 3\\8 & 6\end{bmatrix}$$



$$A^TA = \begin{bmatrix}4 & 8\\3 & 6\end{bmatrix}\begin{bmatrix}4 & 3\\8 & 6\end{bmatrix} = \begin{bmatrix}80 & 60\\60 & 45\end{bmatrix} \qquad \lambda_1 = 125, \quad \lambda_2 = 0$$

$$\underbrace{\begin{bmatrix}4 & 3\\8 & 6\end{bmatrix}}_{A} = \underbrace{\begin{bmatrix}1/\sqrt{5} & 2/\sqrt{5}\\2/\sqrt{5} & -1/\sqrt{5}\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}\sqrt{125} & 0\\0 & 0\end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix}.8 & .6\\.6 & -.8\end{bmatrix}}_{V^T}$$

Almost entirely by inspection!