$\boxed{7}$ Change of basis

Big idea: $A = M^{-1} B M$

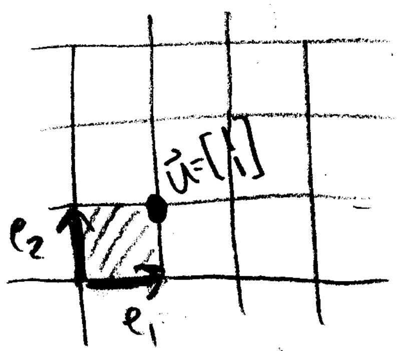
"change of basis" matrix

Example: Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ wrt input basis $v_1 = e_1$, $v_2 = e_2$
output basis $w_1 = e_1$, $w_2 = e_2$

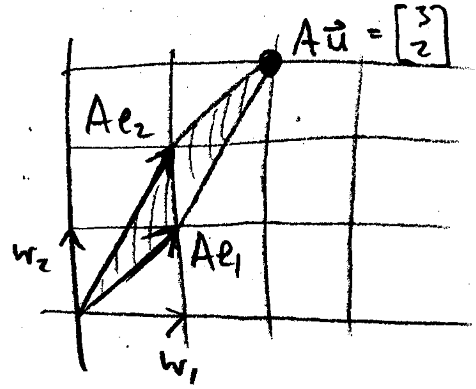
standard input basis

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Say $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$\xrightarrow{A}$

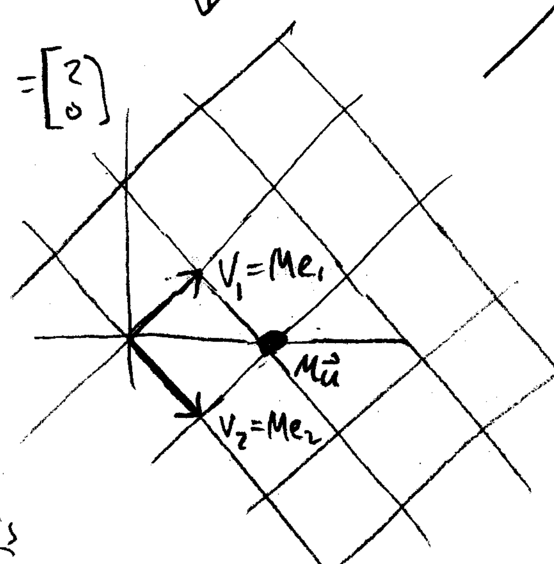


Standard output basis $w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

M

$A M^{-1}$

$Mu = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$



new input basis

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

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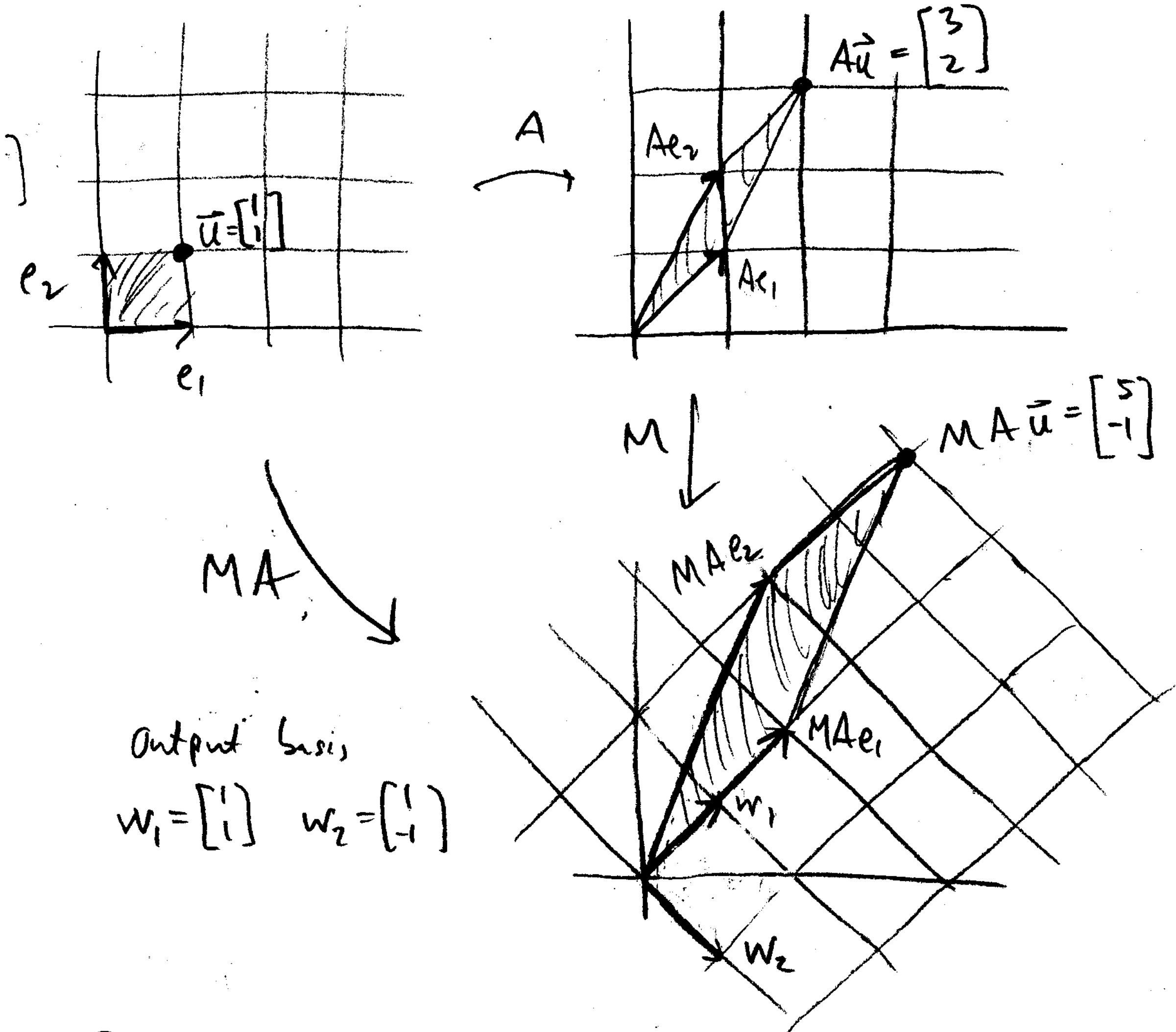
Now, let's see what happens if we change the output basis instead.

Input basis
$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

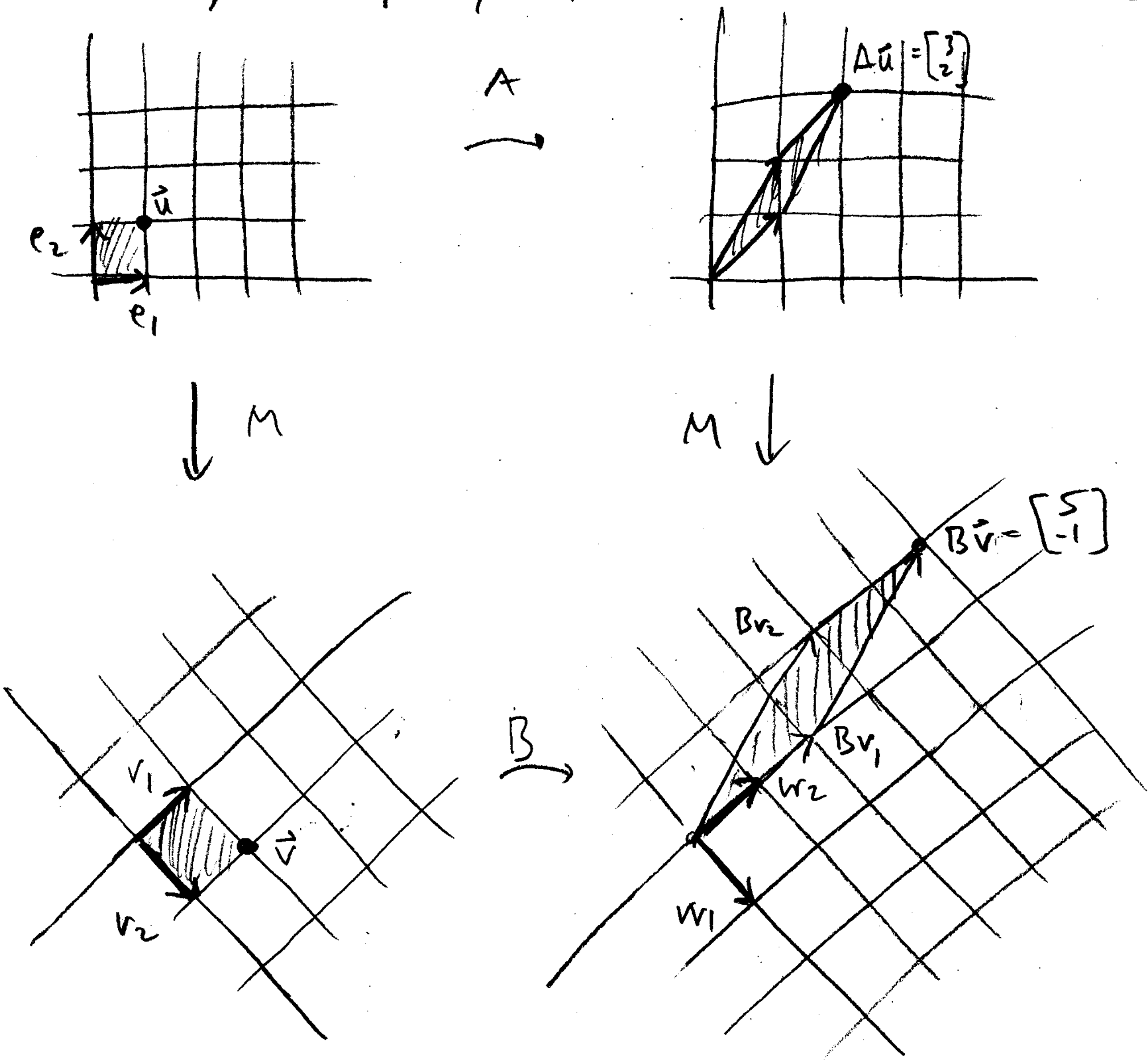


Output basis
$w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $w_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$MA\vec{u} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

Also, $MA\vec{e}_1 = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $MA\vec{e}_2 = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

* Now, let's change both input & output bases: $v_1 = w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = w_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

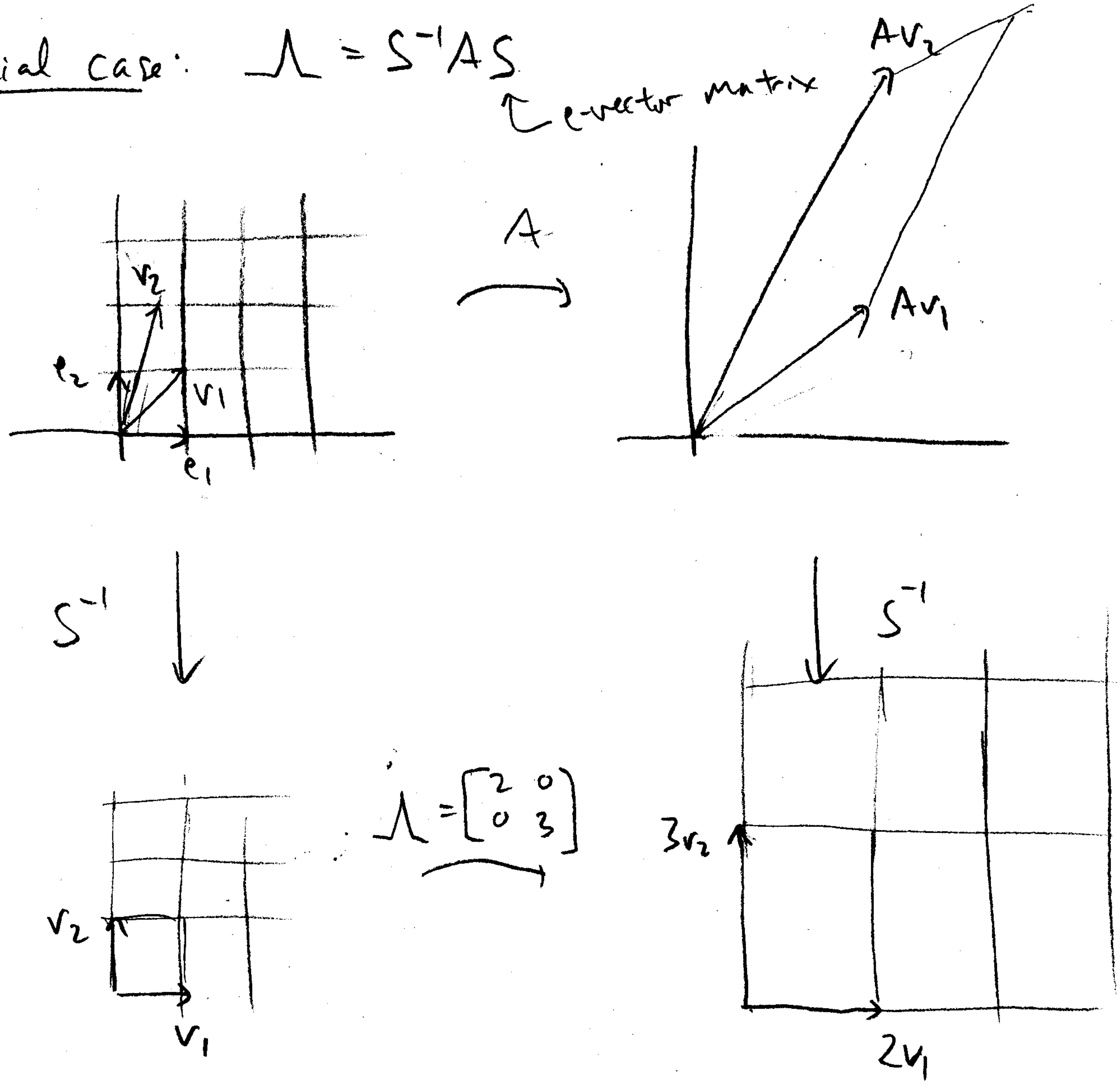


We have $A = M^{-1} B M$

or $B = M A M^{-1}$

<u>Special case</u>: $\Lambda = S^{-1} A S$

$\uparrow$ e-vector matrix



S^{-1}: $v_1 \longmapsto e_1$

$v_2 \longmapsto e_2$