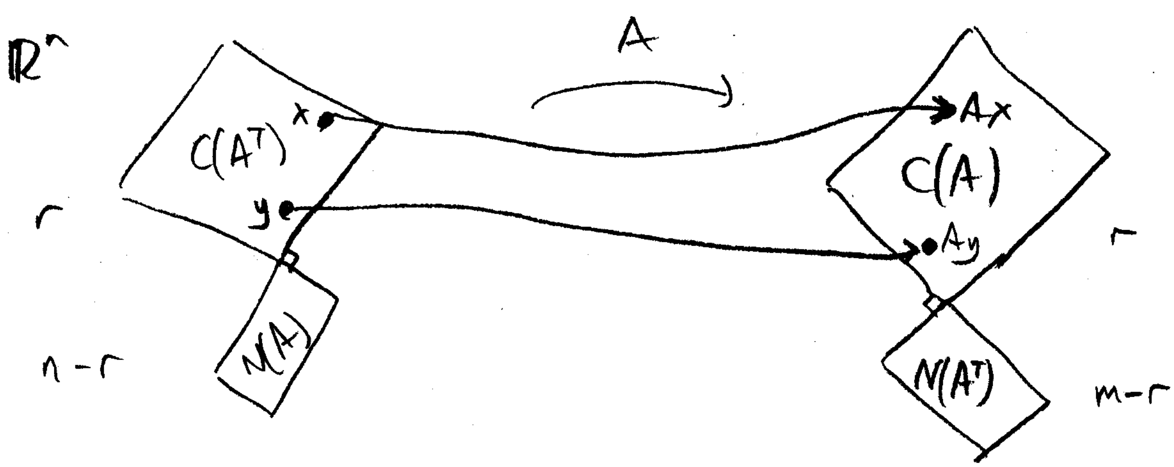
8 Left, right, & pseudoinverses



<u>Claim</u>: A is a <u>one-to-one</u> map from $C(A^T) \longrightarrow C(A)$.

[i.e., if $x \neq y$ in $C(A^T)$, then $Ax \neq Ay$ in $C(A)$.]

<u>Proof</u>. Suppose $Ax = Ay$

$\Rightarrow A(x-y) = 0$

$\Rightarrow x-y$ in $N(A)$

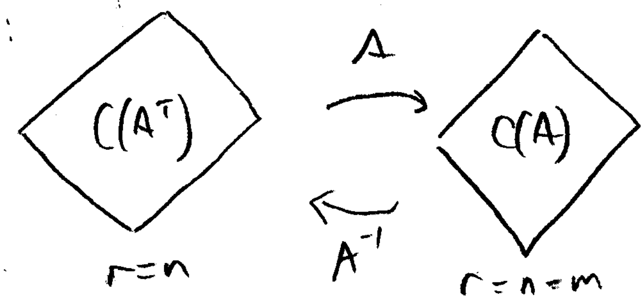
$x-y$ also is in $C(A^T)$ since x, y in $C(A^T)$

$\Rightarrow x-y = 0$ since $N(A) \perp C(A^T)$.

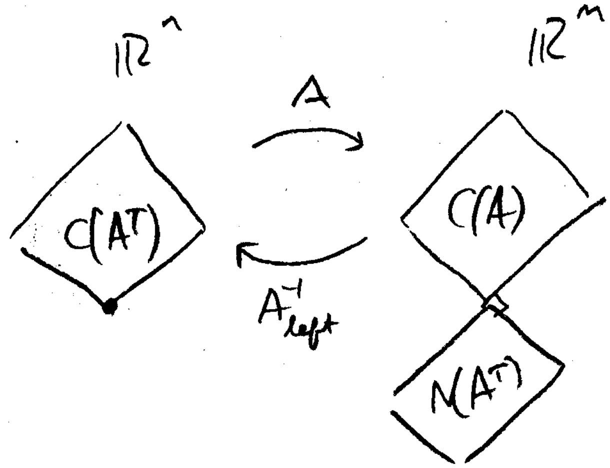
<u>Case 1</u>: A has a <u>2-sided inverse</u>:

$AA^{-1} = I = A^{-1}A$

Full rank $\boxed{r = n = m}$



Case 2: A has a left-inverse



Full column rank $\boxed{r = n < m}$

Nullspace $N(A) = \{0\}$

$Ax = b$ has 0 or 1 sol'n.

A^TA is invertible.

$$\underbrace{(A^TA)^{-1}A^T}A = I_{n\times n}$$

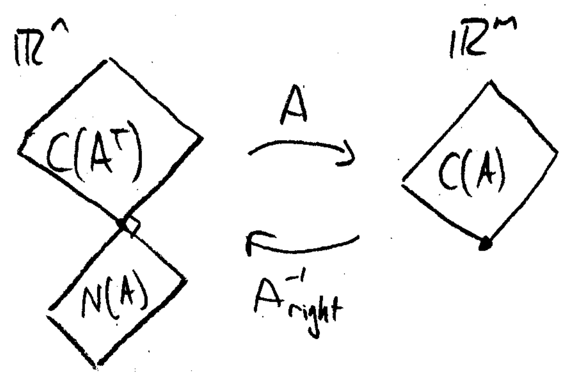
$$A^{-1}_{left} \cdot A = I$$

Reverse order:

$$AA^{-1}_{left} = A\left[(A^TA)^{-1}A^T\right]$$

projection onto $C(A)$!

Case 3: A has a right-inverse



Full row rank $\boxed{r = m < n}$

Left nullspace $N(A^T) = \{0\}$

$Ax = b$ has ∞ sol'ns.

AA^T is invertible.

$$A\underbrace{A^T(AA^T)^{-1}} = I_{m\times m}$$

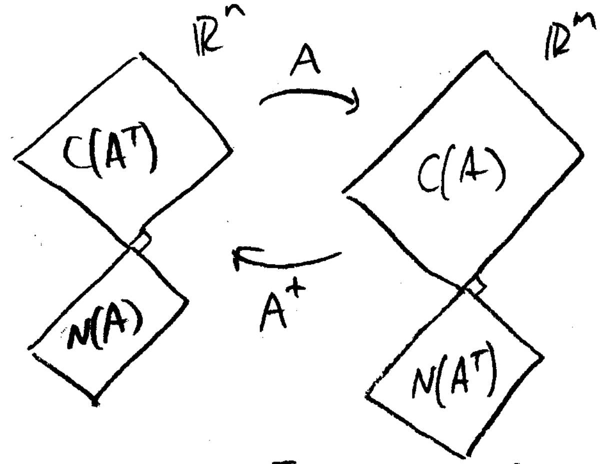
$$A \cdot A^{-1}_{right} = I$$

Reverse order

$$A^{-1}_{right} \cdot A = \left[A^T(AA^T)^{-1}\right]A$$

projection onto $C(A^T)$!

Case 4 The "general case" (any A!)



Recall: $A : C(A^T) \to C(A)$

is invertible!

Goal: Find a matrix A^+

such that $A^+A = \begin{bmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{bmatrix}_{n\times n}$ $\quad AA^+ = \begin{bmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{bmatrix}_{m\times m}$

So, A^+A is the identity map on $C(A^T)$

AA^+ is the identity map on $C(A)$

How to find A^+ (the "pseudoinverse" of A)

Write $A = U\Sigma V^T$ $\quad \Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & 0 & \\ & & \sigma_r & & \\ & 0 & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}, \quad \Sigma^+ = \begin{bmatrix} \sigma_1^{-1} & & & & \\ & \ddots & & 0 & \\ & & \sigma_r^{-1} & & \\ & 0 & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$

$$\boxed{A^+ = V\Sigma^+U^T}$$

$$AA^+ = (U\Sigma V^T)(V\Sigma^+U^T) = U\Sigma\Sigma^+U^T = \Sigma\Sigma^+ = \begin{bmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{bmatrix}_{m\times m}$$

$$A^+A = (V\Sigma^+U^T)(U\Sigma V^T) = V\Sigma^+\Sigma V^T = \Sigma^+\Sigma = \begin{bmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{bmatrix}_{n\times n}$$