Claim: \( A \) is a one-to-one map from \( C(A^T) \to C(A) \).

[i.e., if \( x \neq y \) in \( C(A^T) \), then \( Ax \neq Ay \) in \( C(A) \).]

Proof: Suppose \( Ax = Ay \)

\[ \Rightarrow A(x-y) = 0 \]

\[ \Rightarrow x-y \text{ in } N(A) \]

\( x-y \) also \( \in \) \( C(A^T) \) since \( x, y \) \( \in \) \( C(A^T) \)

\[ \Rightarrow x-y = 0 \] since \( N(A) \perp C(A^T) \).

Case 1: \( A \) has a 2-sided inverse:

\( AA^{-1} = I = A^T A \)

Full rank \( r = n = m \)
Case 2: $A$ has a left-inverse

Full column rank \( r = n < m \)

Nullspace \( N(A^T) = \{0\} \)

\( Ax = b \) has $0$ or $1$ solution.

\( A^T A \) is invertible.

\[
(A^T A)^{-1} A^T A = I_{nxn}
\]

\( A^{-1}_{\text{left}} A = I \)

Case 3: $A$ has a right-inverse

Full row rank \( r = m < n \)

Left nullspace \( N(A^T) = \{0\} \)

\( Ax = b \) has $\infty$ solutions.

\( A A^T \) is invertible.

\[
A A^T (A A^T)^{-1} = I_{m \times m}
\]

\( A A^{-1}_{\text{right}} = I \)

Reverse order:

\( A A^{-1}_{\text{left}} = A [A^T (A A^T)^{-1}] A \)

projection onto \( C(A) \)!
Case 4 The "general case" (any A!)

Recall: $A : C(A^T) \rightarrow C(A)$ is invertible!

Goal: Find a matrix $A^+$ such that $A^+A = \begin{bmatrix} I_{r \times r} & 0 \\ 0 & 0 \end{bmatrix}_{n \times n}$  $AA^+ = \begin{bmatrix} I_{r \times r} & 0 \\ 0 & 0 \end{bmatrix}_{m \times m}$

So, $A^+A$ is the identity map on $C(A^T)$  
$AA^+$ is the identity map on $C(A)$

How to Find $A^+$ (the "pseudo-inverse" of $A$)

Write $A = U \Sigma V^T$  
$\Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix}$, $\Sigma^+ = \begin{bmatrix} \sigma_1^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r^{-1} \end{bmatrix}$

$A^+ = V \Sigma^+ U^T$

$A A^+ = (U \Sigma V^T)(V \Sigma^+ U^T) = U \Sigma \Sigma^+ U^T = \Sigma \Sigma^+ = \begin{bmatrix} I_{r \times r} & 0 \\ 0 & 0 \end{bmatrix}_{m \times m}$

$A^+ A = (V \Sigma^+ U^T)(U \Sigma V^T) = V \Sigma^+ \Sigma V^T = \Sigma^+ \Sigma = \begin{bmatrix} I_{r \times r} & 0 \\ 0 & 0 \end{bmatrix}_{n \times n}$