Chapter 2: What do groups look like?

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A road map for the Rubik's Cube

There are several solution techniques for the Rubik's Cube. If you do a quick Google search, you'll find several methods for solving the puzzle.

These methods describe a sequence of moves to apply relative to some starting position. In many situations, there may be a shorter sequence of moves that would get you to the solution.

In fact, it was shown in July 2010 that every configuration is at most 20 moves away from the solved position!

Let's pretend for a moment that we were interested in writing a complete solutions manual for the Rubik's Cube.

Let me be more specific about what I mean.

A road map for the Rubik's Cube

We'd like our solutions manual to have the following properties:

1. Given any scrambled configuration of the cube, there is a unique page in the manual corresponding to that configuration.
2. There is a method for looking up any particular configuration. (The details of how to do this are unimportant.)
3. Along with each configuration, a list of available moves is included. In each case, the page number for the outcome of each move is included, along with information about whether the corresponding move takes us closer to or farther from the solution.

Let's call our solutions manual the *Big Book*. See Figure 2.1 on page 13 for a picture of what a page in the *Big Book* might look like.

A road map for the Rubik's Cube

We can think of the *Big Book* as a road map for the Rubik's Cube. Each page says, "you are here" and "if you follow this road, you'll end up over there."

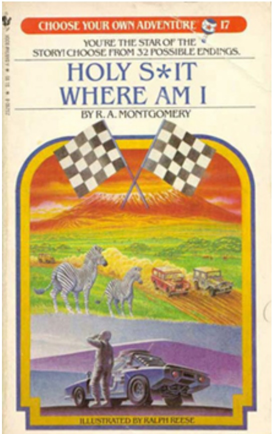


Figure: Potential cover and alternative title for the *Big Book*

Unlike a vintage *Choose Your Own Adventure* book, you'll additionally know whether "over there" is where you want to go or not.

Pros of the *Big Book*:

- We can solve any scrambled Rubik's Cube.
- Given any configuration, every possible sequence of moves for solving the cube is listed in the book (long sequences and short sequences).
- The *Big Book* contains complete data on the moves in the Rubik's Cube universe and how they combine.

Cons of the *Big Book*:

- We just took all the fun out of the Rubik's Cube.
- If we had such a book, using it would be fairly cumbersome.
- We can't actually make such a book. Rubik's Cube has more than 4.3×10^{19} configurations. The paper required to write the book would cover the Earth many times over. The book would require over a billion terabytes of data to store electronically, and no computer in existence can store that much data.

Despite the *Big Book*'s apparent shortcomings, it made for a good thought experiment.

The most important thing to get out of this discussion is that the *Big Book* is a map of a group.

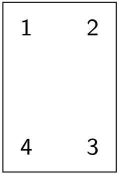
We shall not abandon the mapmaking ideas introduced by our discussion of the *Big Book* simply because the map is too large.

We can use the same ideas to map out any group. In fact, we shall frequently do exactly that.

Let's try something simpler. . .

The Rectangle Puzzle

- Take a blank sheet of paper (our rectangle) and label as follows:



  This is the solved state of our puzzle.

- The idea of the game is to scramble the puzzle and then find a way to return the rectangle to its solved state.

- We are allowed two moves: horizontal flip and vertical flip, where "horizontal" and "vertical" refer to the motion of your hands, rather than any reference to an axis of reflection.

The Rectangle Puzzle

We'll spend some time in Chapter 3 discussing why these two moves and not others are the ones that make sense for this game.

However, it is worth pointing out that these two moves preserve the "footprint" of the rectangle.

Are there any others that preserve its footprint?

Using only the two valid moves, scramble your rectangle. Any sequence of horizontal and vertical flips will do, but don't do any other types of moves.

Now, again using only our two valid moves, try to return your rectangle to the solved position.

Observations?

Question

Do the moves of the Rectangle Puzzle form a group? How can we check?

For reference, here are the rules of a group:

Rule 1.5

There is a predefined list of actions that never changes.

Rule 1.6

Every action is reversible.

Rule 1.7

Every action is deterministic.

Rule 1.8

Any sequence of consecutive actions is also an action.

Road map for the Rectangle Puzzle

Let's make a road map for our newly found group.

Take out a sheet of paper and draw all 4 configurations of the rectangle (do this now!).

Connect the configurations with lines of different color or type.

We've just created our first road map of a group! Observations? What sorts of things does the map tell us about the group?

We see that:

- the group has two generators: horizontal flip and vertical flip. Each generator is represented by the two different types of arrows;
- the group has 4 actions: the "*identity*" action, horizontal flip, vertical flip, and 180° rotation ($r = h \circ v = v \circ h$);
- the map shows us how to get from any one configuration to any other (there may be more than one way to follow the arrows).

Cayley diagrams

It is important to note that how we choose to layout our map is irrelevant.

What is important is that the connections between the various states are preserved.

However, we will attempt to construct our maps in a pleasing to the eye and symmetrical way.

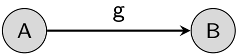
The official name of the type of group road map that we have just created is Cayley diagram, named after the 19th century British mathematician Arthur Cayley.

In general, a Cayley diagram consists of nodes that are connected by colored (or labeled) arrows, where

- an arrow of a particular color represents a specific generator;
- each action of the group is represented by a unique node (sometimes we will label nodes by the corresponding action);
- all necessary arrows are present (more on this later).

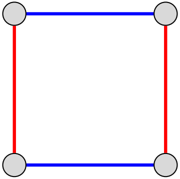
More on arrows

- An arrow corresponding to the generator g from node A to node B means that node B is the result of applying the action g to node A:



- If the reverse of applying generator g is the same as g (this happens with horizontal and vertical flips), then we have a 2-way arrow. Our convention will be to drop the tips of the arrows on all 2-way arrows.

Here is one possible Cayley diagram for our Rectangle Puzzle:



An alternative set of generators for the Retangle Puzzle.

Observe that horizontal flip and a 180° rotation also generate the Rectangle Puzzle group.

Let's build a Cayley graph using these generators. What do you notice about the structure of this alternate Cayley graph?

Both Cayley graphs have the same structure!

Perhaps surprisingly, this might *not* always be the case.

Indeed, there are (more complicated) groups for which different generating sets yield Cayley graphs that are structurally different. We'll see examples of this shortly.

The 2-Light Switch Group

Let's map out another group, which we'll call the *2-Light Switch Group*:

- Consider two light switches side by side that both start in the off position.
- We are allowed 2 actions: flip L switch and flip R switch

Do these actions generate a group?

In small groups, map out the 2-Light Switch Group just like we did for the Rectangle Puzzle. (I suggest using U and D to denote "light switch up" and "light switch down", respectively.)

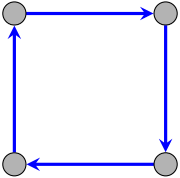
Now, draw the more abstract version of the Cayley diagram. What do you notice?

What we should notice is that the Cayley diagram for the Rectangle Puzzle and the Cayley diagram for the 2-Light Switch Group are essentially the same. The 4 rectangle configurations correspond to the 4 light switch configurations. Horizontal flip and vertical flip correspond to flip L switch and flip R switch.

Although these 2 groups are superficially different, the Cayley diagrams help us see that *they have the same structure*. (The fancy phrase for this phenomenon is that the "two groups are isomorphic"; more on this later.)

Any group with the same Cayley diagram as the Rectangle Puzzle and the 2-Light Switch Group is called the Klein 4-group, and is denoted by V_4 for *vierergruppe*, "four-group" in German. It is named after the mathematician Felix Klein.

It is important to point out that the number of different types (i.e., colors) of arrows is important. For example, the following Cayley diagram *does not* represent V_4.



Think: What group has a Cayley graph like the diagram above?

Question

Is is possible for two groups to have different looking Cayley diagrams yet really be the "same"? (We'll talk more about what "same" means later.)

More Group Exercises

Let's explore a few more examples.

1. In groups of 2–3 (try to mix the groups up again), complete the following exercises (not collected):
   - Exercise 2.1 (see Bob)
   - Exercise 2.3 (see Bob)
   - Exercise 2.5
   - Exercise 2.8 (see Bob)
   - Exercise 2.10
   - Exercise 2.13 (see Bob)
2. I'd like each group to present their solution to one of the problems above.
3. Now, complete Exercise 2.18. I want each group to write up a complete solution.

Properties of Cayley graphs

Observe that at every node of a Cayley graph, there is exactly one out-going edge of each color.

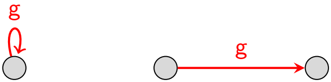
Question 1

Can an edge in a Cayley graph every connect a node to itself?

Question 2

Suppose we have an edge corresponding to generator g that connects a node x to itself. Does that mean that the edge g connects *every* node to itself? In other words, can an action be the *identity action* when applied to some actions but not to others?

Visually, we're asking if the following scenerio can ever occur in a Cayley diagram:



A Theorem and Proof!

Perhaps surprisingly, the previous situation is *impossible*! Let's properly formulate and prove this.

Theorem

Suppose an action g has the property that $gx = x$ for some other action x. Then g is the *identity action*, i.e., $gh = h = hg$ for all other actions h.

Proof

The identity action (we'll denote by 1) is simply the action hh^{-1}, for any action h.

If $gx = x$, then multipling by x^{-1} *on the right* yields:

$$g = gxx^{-1} = xx^{-1} = 1.$$

Thus g is the identity action. □

This was our first mathematical proof! It shows how we can deduce interesting properties about groups *from* the rules, which were not explicitly *built into* the rules.