1. Define a function on $\mathbb{N}$ by setting $d(x,y)=0$ if $x=y$, and $d(x,y)=3^{-k}$ otherwise, where 3^k is the highest power of 3 dividing $|x-y|$. Prove that $(\mathbb{N},d)$ is a metric space.

2. Fix x_0 in a metric space (X,d) and define $\rho\colon X\to\mathbb{R}$ by $\rho(x)=d(x,x_0)$. Show that ρ is continuous.

3. Let $X=\mathbb{R}$, and let $d(x,y)=|x-y|$ be the usual Euclidean metric, and let $d'\colon X\times X\to\mathbb{R}$ be the *discrete metric*, defined by

$$d'(x,y)=\begin{cases}0 & x=y\\ 1 & x\neq y.\end{cases}$$

   (a) Prove or disprove: the identity map $i\colon(\mathbb{R},d)\to(\mathbb{R},d')$ is continuous.

   (b) Prove or disprove: the identity map $i'\colon(\mathbb{R},d')\to(\mathbb{R},d)$ is continuous.

   Use the (ϵ,δ)-definition of continuity. The "inverse image of open sets" characterization, which we have not seen yet, will almost trivialize this question.

4. Let (X,d), (Y,d'), and (Z,d'') be metric spaces. A function $f\colon X\to Y$ is *uniformly continuous* if for all $\epsilon>0$, there is some $\delta>0$ such that $d(x,y)<\delta$ implies $d'(f(x),f(y))<\epsilon$. This is a stronger condition than ordinary continuity because δ does not depend on $x\in X$.

   (a) Give an example of a function that is continuous but not uniformly continuous.

   (b) Prove that if $f\colon X\to Y$ and $g\colon Y\to Z$ are uniformly continous, then $g\circ f\colon X\to Z$ is uniformly continuous.