Throughout, a metric space X is endowed with a distance function $d\colon X \times X \to \mathbb{R}$, even if not explicitly mentioned. Also, if the distance function for $\mathbb{R}^k$ is not specified, it is assumed to be Euclidean.

1. Let a and b be distinct points in a metric space X. Prove that there are neighborhoods N_a and N_b of a and b respectively such that $N_a \cap N_b = \emptyset$. A topological space with this property is said to be *Hausdorff*; thus this exercise shows that metric spaces are Hausdorff.

2. Let (X, d) be a metric space. The distance between a point $x \in X$ and a nonempty set $A \subseteq X$ is defined by
$$d(x, A) = \inf\{d(x, a) \mid a \in A\}\,.$$
   (a) Prove that $d(x, A) \leq d(x, y) + d(y, A)$.
   (b) Prove that the function $f\colon X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is continuous.
   (c) Prove that $d(x, A) = 0$ if and only if every neighborhood of x contains some point in A.

3. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$. What are the answers to these questions in $\mathbb{R}^2$ under the discrete metric?

4. Let (X, d_1) and (Y, d_2) be metric spaces and $f\colon X \to Y$ be continuous. Define a distance function d on $X \times Y$ in the standard manner:
$$d((x, y), (x', y')) = \max\{d_1(x, x'), d_2(y, y')\}\,.$$
   (a) Prove that the graph $\Gamma_f := \{(x, f(x)) \mid x \in X\}$ is a closed subset of $(X \times Y, d)$.
   (b) Show by example that a function whose graph is closed need not be continuous.