Throughout, a metric space X is endowed with a distance function $d\colon X \times X \to \mathbb{R}$, even if not explicitly mentioned. Also, if the distance function for $\mathbb{R}^k$ is not specified, it is assumed to be Euclidean.

1. Let A_1, A_2, A_3,... be subsets of a metric space (X, d). The *closure* of A_i, denoted $\bar{A}_i$, is defined to be the union of A_i with its limit points. It is elementary to show that the closure of a set is indeed closed.

   (a) If $B_n = \bigcup_{i=1}^{n} A_i$, prove that $\bar{B}_n = \bigcup_{i=1}^{n} \bar{A}_i$, for $n = 1, 2, 3, \dots$.

   (b) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\bar{B} \supseteq \bigcup_{i=1}^{\infty} \bar{A}_i$. Show, by an example, that this inclusion can be proper.

2. Let E° denote the set of all interior points of a set $E \subseteq X$, which we call the *interior* of E.

   (a) Prove that E° is always open.

   (b) Prove that E is open if and only if $E^\circ = E$.

   (c) If $U \subseteq E$ and U is open, prove that $U \subseteq E^\circ$.

   (d) Prove that the complement of E° is the closure of the complement of E.

   (e) Do E and $\bar{E}$ always have the same interiors? Prove or disprove.

   (f) Do E and E° always have the same closures? Prove or disprove.

3. Let (Y, d_Y) be a subspace of the metric space (X, d_X).

   (a) Prove that a subset $V \subseteq Y$ is an open set of Y if and only if there is an open subset U of X such that $V = Y \cap U$.

   (b) Prove that a subset $G \subseteq Y$ is a closed set of Y if and only if there is an closed subset F of X such that $G = Y \cap F$.

   (c) For a point $y \in Y$, prove that a subset $N \subseteq Y$ is a neighborhood of y if and only if there is a neighborhood M of y in X such that $N = Y \cap M$.

4. Two metric spaces (X, d_X) and (Y, d_Y) are *topologically equivalent* if there are inverse functions $f\colon X \to Y$ and $g\colon Y \to X$ that are continuous.

   (a) Prove that the open interval $(-\pi/2, \pi/2)$ is topologically equivalent to $\mathbb{R}$.

   (b) Prove that any two open intervals are topologically equivalent to each other, and hence to $\mathbb{R}$.