1. Suppose $(X, \tau_X)$ and $(Y, \tau_Y)$ are topological spaces and $A \subseteq X$ and $B \subseteq Y$. Prove that:

   (a) $\mathrm{Int}(A \times B) = \mathrm{Int}(A) \times \mathrm{Int}(B)$;

   (b) $\overline{A \times B} = \overline{A} \times \overline{B}$;

   (c) $\partial(A \times B) = [\partial A \times \overline{B}] \cup [\overline{A} \times \partial B]$.

2. Let $X_1, X_2$ be sets and let $\pi_i \colon X_1 \times X_2 \to X_i$ denote the standard projection maps for $i = 1, 2$. The Cartesian product $X_1 \times X_2$ can be thought of as the "smallest" set that projects onto both $X_1$ and $X_2$. This can be formalized by the following "universal property":

   For any set $Z$ along with maps $f_i \colon Z \to X_i$, there is a unique map $f \colon Z \to X_1 \times X_2$ satisfying $\pi_i \circ f = f_i$ for $i = 1, 2$. That is, the following diagram commutes:

$$\begin{tikzcd} & Z \arrow[dl, "f_1"'] \arrow[d, dashed, "\exists! f"] \arrow[dr, "f_2"] & \\ X_1 & X_1 \times X_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X_2 \end{tikzcd}$$

   Prove this.

3. Let $(X_1, \tau_1), \dots, (X_n, \tau_n)$ be topological spaces and endow $X := \prod_{i=1}^n X_i$ with the product topology $\tau_X$. Let $\pi_i \colon X \to X_i$ be the standard projection maps; recall these are continuous.

   (a) Suppose $f \colon Z \to X$ is a function from a topological space $(Z, \tau_Z)$. Show that $f$ is continuous if and only if each $f_i := \pi_i \circ f$ is continuous:

$$\begin{tikzcd} X \arrow[r, "\pi_i"] & X_i \\ Z \arrow[u, "f"] \arrow[ur, "\pi_i \circ f"'] & \end{tikzcd}$$

   (b) Prove the following "universal property" of the product topology: For any collection of continuous maps $f_i \colon Z \to X_i$, there exists a unique continuous map $f \colon Z \to X$ such that $f_i = \pi_i \circ f$ for all $i$. That is, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\pi_i"] & X_i \\ Z \arrow[u, dashed, "f"] \arrow[ur, "f_i"'] & \end{tikzcd}$$