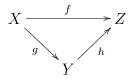
Throughout, a topological space X is endowed with a topology τ , even if not explicitly mentioned.

- 1. A topological space is *normal* if for any two disjoint closed sets F_1, F_2 , there exists disjoint open sets U_1, U_2 such that $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$. Suppose X is Hausdorff and normal and let F be a closed set. Define an equivalence relation on X by $x \sim y$ whenever both x and y belong to F. Prove that the quotient space X/\sim is Hausdorff.
- 2. (a) Let $p: X \to Y$ be a continuous map. Show that if there is a continuous map $f: Y \to X$ such that $p \circ f = 1_Y$, the identity map on Y, then p is a quotient map.
 - (b) If $A \subset X$, a retraction of X onto A is a continuous map $r: X \to A$ such that r(a) = a for each $a \in A$. Show that a retraction is a quotient map.
- 3. A map $f: X \to Y$ of topological spaces is called an *open (closed) map* if the image of every open (closed) set in X is again open (closed). Show that if a continuous surjective map $f: X \to Y$ is either open or closed, then it is a quotient map.
- 4. Consider the following commutative diagram, i.e., $h \circ g = f$.



Prove that if f and g are quotient maps and h is a bijection, then h must be a homeomorphism.

- 5. Use the previous problem(s) to establish the following homeomorphisms. Prove all of your claims.
 - (a) Let X = [0, 1] and let \sim be the equivalence relation that such that $x \sim x$ and $0 \sim 1$. Prove that X/\sim is homeomorphic to S^1 . [*Hint*: Can you define a continuous surjective map $f: [0, 1] \rightarrow S^1$?]
 - (b) Consider the disjoint union $X = D_1 \sqcup D_2$ of two unit disks. Identify $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 = x_2$ and $y_1 = y_2$ where $(x_1, y_1) \in \partial D_1$ and $(x_2, y_2) \in \partial D_2$. Show that X/\sim is homeomorphic to the unit 2-sphere S^2 in \mathbb{R}^3 .
 - (c) Consider the unit square $X := [0,1] \times [0,1]$ in \mathbb{R}^2 . Identify $(0,y) \sim (1,y)$ and $(x,0) \sim (x,1)$ for all $x, y \in [0,1]$ The quotient space X/\sim is called a *torus*. Show that it is homeomorphic to $S^1 \times S^1$.