Throughout, a topological space X is endowed with a topology τ, even if not explicitly mentioned.

1. A topological space is *normal* if for any two disjoint closed sets F_1, F_2, there exists disjoint open sets U_1, U_2 such that $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$. Suppose X is Hausdorff and normal and let F be a closed set. Define an equivalence relation on X by $x \sim y$ whenever both x and y belong to F. Prove that the quotient space $X/\sim$ is Hausdorff.

2. (a) Let $p\colon X \to Y$ be a continuous map. Show that if there is a continuous map $f\colon Y \to X$ such that $p \circ f = 1_Y$, the identity map on Y, then p is a quotient map.

   (b) If $A \subset X$, a *retraction* of X onto A is a continuous map $r\colon X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

3. A map $f\colon X \to Y$ of topological spaces is called an *open (closed) map* if the image of every open (closed) set in X is again open (closed). Show that if a continuous surjective map $f\colon X \to Y$ is either open or closed, then it is a quotient map.

4. Consider the following commutative diagram, i.e., $h \circ g = f$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Z \\ & Y \arrow[ur, "h"'] & \end{tikzcd}$$

   Prove that if f and g are quotient maps and h is a bijection, then h must be a homeomorphism.

5. Use the previous problem(s) to establish the following homeomorphisms. Prove all of your claims.

   (a) Let $X = [0, 1]$ and let $\sim$ be the equivalence relation that such that $x \sim x$ and $0 \sim 1$. Prove that $X/\sim$ is homeomorphic to S^1. [*Hint*: Can you define a continuous surjective map $f\colon [0, 1] \to S^1$?]

   (b) Consider the disjoint union $X = D_1 \sqcup D_2$ of two unit disks. Identify $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 = x_2$ and $y_1 = y_2$ where $(x_1, y_1) \in \partial D_1$ and $(x_2, y_2) \in \partial D_2$. Show that $X/\sim$ is homeomorphic to the unit 2-sphere S^2 in $\mathbb{R}^3$.

   (c) Consider the unit square $X := [0, 1] \times [0, 1]$ in $\mathbb{R}^2$. Identify $(0, y) \sim (1, y)$ and $(x, 0) \sim (x, 1)$ for all $x, y \in [0, 1]$ The quotient space $X/\sim$ is called a *torus*. Show that it is homeomorphic to $S^1 \times S^1$.