Throughout, a topological space X is endowed with a topology τ, even if not explicitly mentioned.

1. A topological space X is *disconnected* if it is the union of two disjoint nonempty open sets. Otherwise it is *connected.*
   (a) Prove that X is disconnected if and only if there exists a continuous surjective map $f\colon X \to \{0,1\}$, where $\{0,1\}$ has the discrete topology.
   (b) Suppose X has the property that every two point subset lies inside a connected subset of X. Prove that X is connected.

2. Two subsets $A, B \subseteq X$ are *separated* in X if $A \cap \overline{B} = \overline{A} \cap B = \emptyset$. A subspace $Y \subseteq X$ is connected if Y is not the union of two separated sets. Prove that the following are equivalent:
   (i) Y is separated.
   (ii) Y is the union of two sets that are open in the subspace topology of Y.
   (iii) Y is the union of two sets that are closed in the subspace topology of Y.

3. Prove that if $A \subseteq X$ is connected, and $A \subseteq B \subseteq \overline{A}$, then B is connected.

4. A topological space X is *step connected* if given any open cover $\mathcal{U}$ of X and any pair of points $x, y \in X$, there is a finite sequence $U_1, \dots, U_n$ of sets belonging to $\mathcal{U}$ so that $x \in U_1$, $y \in U_n$ and $U_i \cap U_{i+1} \neq \emptyset$ of $1 \leq j < n$. Prove that X is step connected if and only if it is connected.

5. A space X is *path connected* if every two points $x, y \in X$ can be connected via a *path.* That is, there is a continuous function $f\colon [a,b] \to X$ with $f(a) = x$, $f(b) = y$, and $f([a,b]) \subseteq X$. Prove or disprove the following.
   (a) X and Y are path connected if and only if $X \times Y$ is path connected.
   (b) If $A \subseteq X$ and A is path connected, then $\overline{A}$ is path connected.
   (c) If $f\colon X \to Y$ is continuous and X is path connected, then $f(X)$ is path connected.
   (d) If $\{A_\alpha\}$ is a collection of path connected subspaces of X that has a nonempty intersection, then $\cup A_\alpha$ is path connected.