Throughout, X is a path-connected topological space. Given a path $\alpha\colon I \to X$ with endpoints $\alpha(0) = x_0$ and $\alpha(1) = x_1$, define the map

$$\hat{\alpha}\colon \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)\,, \qquad \hat{\alpha}([f\,]) = [\bar{\alpha}] * [f\,] * [\alpha]\,.$$

If $h\colon (X, x_0) \longrightarrow (Y, y_0)$ is continuous, then the *induced homomorphism* relative to x_0 is

$$h_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)\,, \qquad h_*([f\,]) = [h \circ f\,]\,.$$

1. Let $\alpha\colon I \to X$ be a path from x_0 to x_1, and $\beta\colon I \to X$ a path from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

2. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

3. Recall that a *retraction* of X onto a subset A is a continuous map $r\colon X \to A$ such that $r \circ r = r$. Equivalently, this means that $ri = 1_A$, where $i\colon A \hookrightarrow X$ is the inclusion map and 1_A is the identity mapping on A.

   (a) Let $a_0 \in A$. Prove that if $r\colon X \to A$ is a retraction, then $i_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is one-to-one and $r_*\colon \pi_1(X, a_0) \to \pi_1(A, a_0)$ is onto.

   (b) Prove that a retract of a contractible space is contractible.

4. Let A be a subspace of $\mathbb{R}^n$ and $h\colon (A, a_0) \longrightarrow (Y, y_0)$ be continuous. Show that if h extends to a continuous map of $\mathbb{R}^n$ into Y (that is, if there is some $f\colon \mathbb{R}^n \to Y$ such that $f \circ i = h$ for the inclusion map $i\colon A \to \mathbb{R}^n$), then h_* is the trivial homomorphism.

5. Show that the induced homomorphism of a continuous map is independent of base point, up to isomorphism. More precisely, let $h\colon X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1, and let $\beta = h \circ \alpha$. Show that

   $$\hat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \hat{\alpha}\,.$$

   In other words, the following diagram commutes:

   $$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "(h_{x_0})_*"] \arrow[d, "\hat{\alpha}"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\beta}"] \\ \pi_1(X, x_1) \arrow[r, "(h_{x_1})_*"] & \pi_1(Y, y_1) \end{tikzcd}$$