Math 2080: Differential Equations
Worksheet 7.1: The heat equation

NAME:

You will solve for the function $u(x,t)$ defined for $0 \leq x \leq \pi$ and $t \geq 0$ which satisfies the following IVP/BVP of the heat equation:

$$u_t = c^2 u_{xx} \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = 4\sin x - 3\sin 2x.$$

(a) Carefully descsribe (and sketch) a physical situation that this models. [Use a computer to sketch the initial condition.]

(b) Assume that $u(x,t) = f(x)g(t)$. Compute u_t and u_{xx}, and derive boundary conditions for $f(x)$.

(c) Plug $u = fg$ back into the PDE and separate variables by dividing both sides of the equation by $c^2 fg$. Now set this equal to a constant λ, and write down two ODEs: one for $g(t)$, and a BVP for $f(x)$.

(d) Recall from Section 6 that the BVP for f has a solution $f_n(x)$ for each $\lambda = -n^2$ where $n = 1, 2, \dots,$ and that solution is $f_n(x) = b_n \sin nx$. Now, given such $\lambda = -n^2$ solve the ODE for g. Call this solution $g_n(t)$.

(e) Find the general solution of the PDE. As before, it will be a superposition (infinite sum) of solutions $u_n(x,t) = f_n(x)g_n(t)$.

(f) Find the particular solution to the IVP by using the initial condition.

(g) What is the steady-state solution? Give a physical interpretation for this quantity.