Math 2080: Differential Equations
Worksheet 7.4: The wave equation

NAME:

Let $u(x,t)$ be defined for $0 \leq x \leq 1$ and $t > 0$, and consider the following PDE

$$u_{tt} + 2\beta u_t = u_{xx}, \qquad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = x(\pi - x), \quad u_t(x,0) = 1.$$

where $0 < \beta < 1$ is a constant. This is the wave equation where the u_t term models transverse vibrations being in a medium that imparts a resistance proportional to the instantaneous velocity.

(a) Describe and sketch this situation at $t = 0$.

(b) Assume that there is a solution of the form $u(x,t) = f(x)g(t)$. Plug this back into the PDE and get an ODE for $g(t)$ and a BVP for $f(x)$.

(c) The BVP should be familiar: $f'' = -\lambda f$, $f(0) = f(\pi) = 0$, and we've seen that $\lambda_n = n^2$ for $n = 0, 1, 2, \ldots$ and $f_n(x) = b_n \sin nx$. Solve the ODE for $g(t)$.

(d) Write down the general solution to this PDE.

(e) Use the initial conditions to find the particular solution solving the boundary and initial conditions.

(f) What is the long-term behavior of this system? Give both a mathematical and physical justification.