

1. Carry out the steps outlined below for the following predator–prey model:

$$\begin{aligned}P_{t+1} &= P_t(1 + .8(1 - P_t)) - 4P_tQ_t \\ Q_{t+1} &= .9Q_t + 2P_tQ_t\end{aligned}$$

- (a) Compute the equilibria.
 - (b) Use MATLAB and the `twopop` program to make an informed guess as to whether the equilibria are stable or unstable. Print out or sketch the phase portrait.
 - (c) Linearize the model at each of the equilibria and compute eigenvalues to determine stability.
2. One approach to preventing disease spread is to simply quarantine infectives. Suppose a disease is modeled by the SIR model, but people who get the disease are health-conscience and quarantine themselves. The net result is that a fraction q of the infectives are prevented from having contacts with the susceptibles. Only $1 - q$ of the infectives will be able to spread the disease.
 - (a) Modify the equations of the SIR model to reflect this. What value of q gives the usual SIR model?
 - (b) Quarantining can be viewed as a way of modifying the transmission coefficient. Suppose an SIR model has transmission coefficient α , and a fraction q of the infectives are successfully quarantined. Then the model with quarantining is identical to a standard SIR model with some other transmission coefficient α' , the *effective transmission coefficient*. Give a formula for α' in terms of α and q .
 - (c) Use the MATLAB program `sir` to investigate the behavior of your quarantine model for $N = 100$, $\alpha = 0.001$, and $\gamma = 0.05$, and vary q from 0 to 1. Explain the qualitative behavior you see. Can you find a value of q that prevents an epidemic from occurring, regardless of I_0 ? Estimate the smallest such q .
 3. Another approach to preventing disease spread is vaccination of susceptibles. Suppose a public health organization offers a vaccine for a disease modeled by the SIR model. One simple model of this situation counts each successful vaccination in the removed class throughout the duration of the model.
 - (a) Suppose that all vaccinations occur before the time $t = 0$. Even if this is not the case, we may assume it is – why?
 - (b) Suppose with $N = 100$, we have $I_0 = 1$, with the removed class composed of the fraction q of the population that was successfully vaccinated. Give formulas for S_0 and R_0 (the initial number of recovered people, *not* the basic reproductive number \mathcal{R}_0). What value of q gives the usual SIR model?
 - (c) Repeat Part (c) from the previous problem for this situation.

4. An isolated island population of 100 individuals is exposed to a particularly deadly disease; an infected individual remains contagious until overcome by death after 4 days. We want to predict the disease's effect on the community on a daily basis. Suppose initially one individual is stricken with the disease.
- What is the removal rate γ ?
 - For what values of the *relative removal rate* $\rho := \gamma/\alpha$ will an epidemic occur? Use this to determine for what values of the transmission coefficient α an epidemic will occur.
 - Use a computer program such as `sir` to estimate the number of days until the epidemic peaks for the values of $\alpha = .003, .005, .01, \text{ and } .0125$, presenting your data in a table. How does the magnitude of α relate to the time until the peak?
 - Calculate the basic reproductive numbers and the relative remove rates ρ for the values of α above, adding that information to your table.
5. The following difference equation is called the *SIS model*:

$$\begin{aligned}\Delta S &= -\alpha SI + \gamma I \\ \Delta I &= \alpha SI - \gamma I.\end{aligned}$$

- What disease might be modeled well by the SIS framework?
- Use a computer program such as `sir` or `twopop` to explore the dynamics of the SIS model. Vary the parameters α, γ, N, S_0 , and I_0 . Describe your findings.
- Solve for all equilibria (S^*, I^*) . Are these biologically reasonable? An equilibrium $I^* > 0$ is called an *endemic equilibrium*. Can an SIS disease be endemic?
- Since $S_t + I_t = N$ is constant, substitute $I_t = N - S_t$ back into the formula for S_t and find a formula for S_{t+1} in terms of S_t . Find a formula for I_{t+1} in terms of I_t .
- For the SIR model, the threshold value $\rho := \gamma/\alpha$, called the *relative remove rate*, plays an important role. What does it represent? Is there an analogous threshold value for the SIS model? If so, find it. If not, explain why.
- For the SIR model, the basic reproductive number \mathcal{R}_0 plays an important role. How should one define \mathcal{R}_0 for the SIS model? Justify your answer.