

# Modeling biochemical reactions

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Math 4500, Fall 2016

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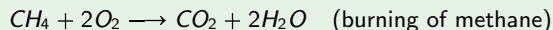
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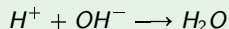
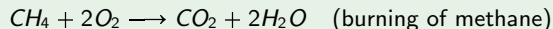
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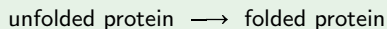
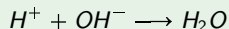
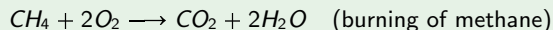
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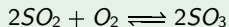
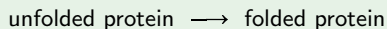
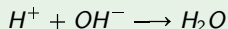
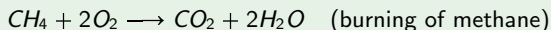
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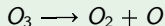
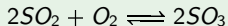
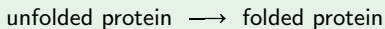
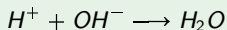
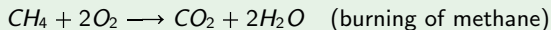
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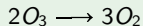
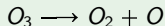
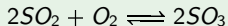
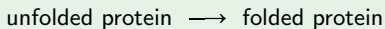
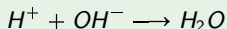
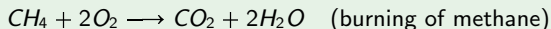
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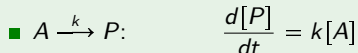
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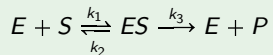
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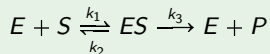


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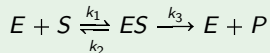
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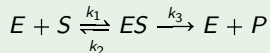
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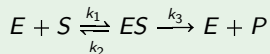
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- $E_0$  is constant.
- Enzyme-substrate complex reaches equilibrium much earlier than the product does, so  $\frac{d[ES]}{dt} \approx 0$ .

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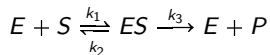
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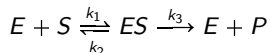
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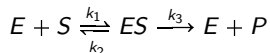
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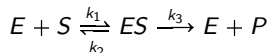
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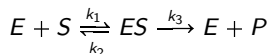
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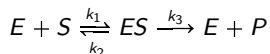
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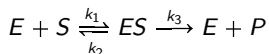
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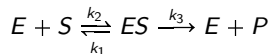
$$\frac{d[P]}{dt} = \frac{V_{\max}[S]}{\underbrace{K_m + [S]}_{f([S])}}, \quad \text{where } V_{\max} = k_3 E_0, \quad \text{and } K_m = \frac{k_2 + k_3}{k_1}$$

### Remarks

- The “reaction rate”,  $f([S])$ , is a strictly increasing function of  $[S]$ .
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## Michaelis–Menten equation

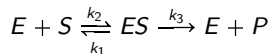
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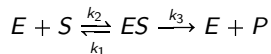


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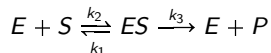
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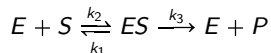
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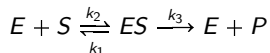
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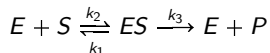
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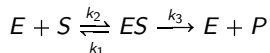
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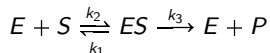
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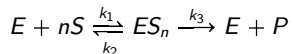
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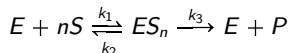
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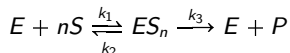
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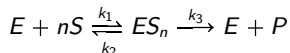
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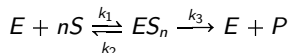
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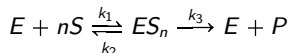
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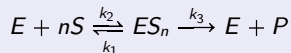
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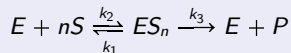
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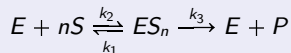
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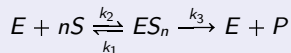
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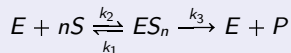
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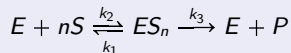
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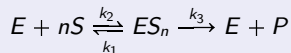
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# Hill equations

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The following shows several “Hill functions”  $y = \frac{t^n}{1 + t^n}$ , for various values of  $n$ .

