Difference Equations

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Math 4500, Mathematical Modeling

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Letting $\lambda = 1 + f - d$ (the "*finite growth rate*"), we can write this as $\boxed{P_{t+1} = \lambda P_t}$.

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It is not difficult to see the closed-form solution $P_t = \lambda^t P_0$. This is called *exponential growth*.

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Exercise

Can you think of a model that is discrete time and discrete space? Or continuous time and discrete space?

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Which of these are more suited for difference equations, and which for differential equations?

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- Suppose the growth rate decreases *linearly* with P.

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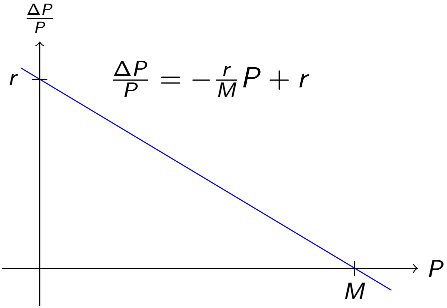

Since the growth rate decreases *linearly* with P, basic algebra gives

$$\frac{\Delta P}{P} = -\frac{r}{M}P + r = r\left(1 - \frac{P}{M}\right).$$

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Substituting $\Delta P = P_{t+1} - P_t$ into $\frac{\Delta P}{P} = r\left(1 - \frac{P}{M}\right)$, followed by easy algebra yields the discrete logistic model:

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Exercise

What is $F(x)$ in the discrete logistic model? [It must satisfy $P_{t+1} = F(P_t)$.]

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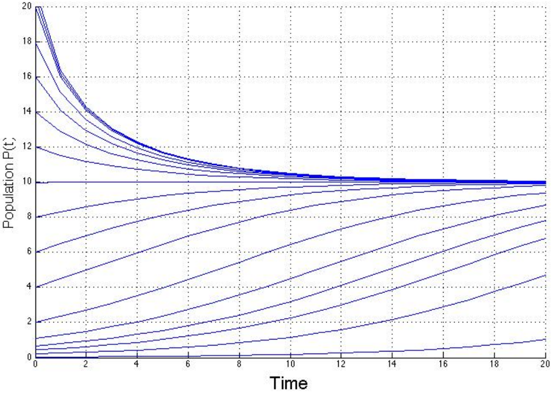
Here are some solutions to the equation $P_{t+1} = P_t + .2P_t(1 - \frac{P_t}{10})$.



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We can numerically find $P_0, P_1, P_2, \ldots$ by plotting $F(x) = x + 0.8x(1 - \frac{x}{10})$ and $y = x$ on the same axes, and then by "cobwebbing":

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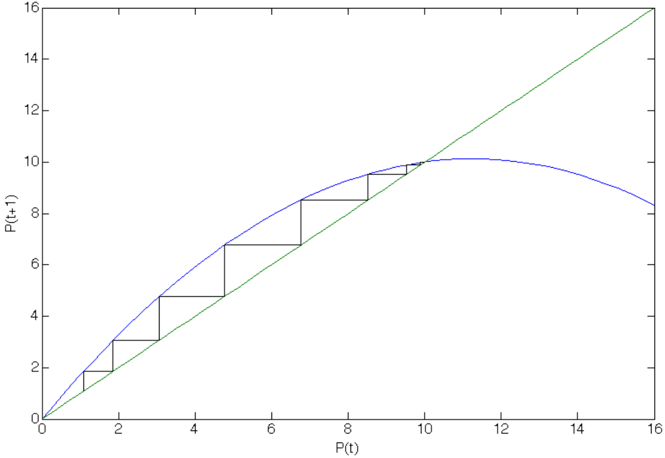

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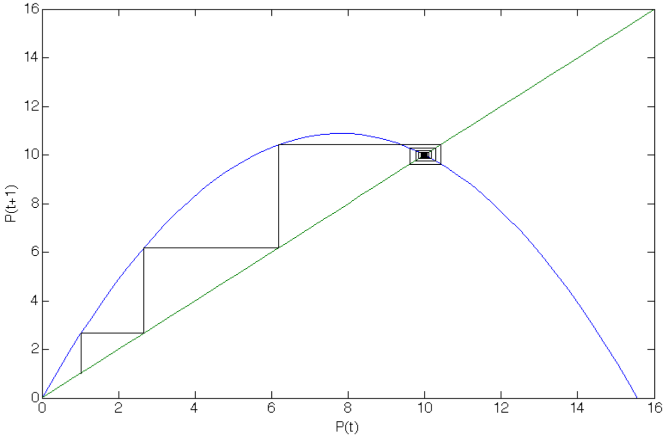

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4. What features about a population are highlighted in the logistic equation using difference equations that do not arise using differential equations?