Analyzing non-linear models

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Long-term behavior

Consider a difference equation $P_{t+1} = F(P_t)$, and the sequence $P_0, P_1, P_2, P_3, \dots$.

Transient behavior (the initial few iterations) will usually "die out."

In contrast, *long-term* behavior is often independent, or "almost independent" of the initial conditions.

Types of periodic cycles

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- $P_{t+1} = P_t$ is a fixed point (or "equilibrium", or "steady-state").
- We can also have longer periodic cycles:

$$\dots, \underbrace{P_t, P_{t+1}, \dots, P_{t+k-1}}_{\text{length-}k\text{ cycle}}, \underbrace{P_{t+k}}_{=P_t}, \underbrace{P_{t+k+1}}_{=P_{t+1}}, \underbrace{P_{t+k+2}}_{=P_{t+2}}, \dots$$

Note that fixed poitns are cycles of length $k = 1$.

Fixed points

Goals

- How to find them.
- How to classify them (stable, unstable, etc.)

There are two ways (which are essentially the same) to find the fixed points:

(i) Set $\Delta P = 0$ and solve for P.

(ii) Set $P_t = P_{t+1} = P^*$ and solve for P^*.

Example. Consider $P_{t+1} = P_t\left(1 + .7\left(1 - \frac{P_t}{10}\right)\right)$.

Set $P_t = P_{t+1} = P^*$ and solve

$$P^* = P^*\left(1 + .7\left(1 - \frac{P^*}{10}\right)\right).$$

Clearly, the fixed points are $P^* = 0$ and $P^* = 10$.

This should be visually obvious from either plots of $P(t)$, or from any cobwebbing plot.

Bifurcation in the logistic map $P_{t+1} = rP_t(1 - P_t)$

Consider the following "normalized" logistic model:

$$P_{t+1} = rP(1 - P).$$

Main idea

Understand how the dynamics (e.g., the cobwebbing diagram) change qualitatively as the parameter r changes.

There are two fixed points:

- $P^* = 0$: $p_{t+1} \approx (1 + r)p_t$, unstable, since $1 + r > 1$.
- $P^* = 1 - \frac{1}{r}$: $p_{t+1} \approx \underbrace{(1 - r)}_{\text{"stretching factor"}} p_t$, stability depends on r.

We'll analyze the nature of fixed point $P^* = 1$ for various values of r.

Bifurcation in the logistic map $P_{t+1} = rP_t(1 - P_t)$

Case I: $0 < r < 1$

Since $0 < 1 - r < 1$, the perturbation is $p_{t+1} \approx \underbrace{(1 - r)}_{>0} p_t$.

Therefore, $p_t \to 0$ without changing sign. This is overdamping.

[GRAPHICS]

Bifurcation in the logistic map $P_{t+1} = rP_t(1 - P_t)$

Case II: $1 < r < 2$

Since $0 < 1 - r < 1$, the perturbation is $p_{t+1} \approx \underbrace{(1 - r)}_{<0} p_t$.

Therefore, $p_t \to 0$, but the sign toggles. This is underdamping.

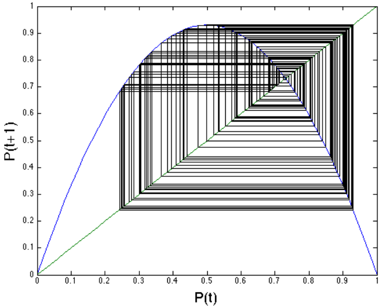
[GRAPHICS]

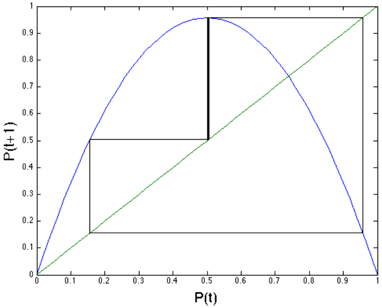
Bifurcation in the logistic map $P_{t+1} = rP_t(1 - P_t)$

Case III: $r > 2$

Since $1 - r < -1$, the perturbation is $p_{t+1} \approx \underbrace{(1-r)}_{||\cdot||>1} p_t$.

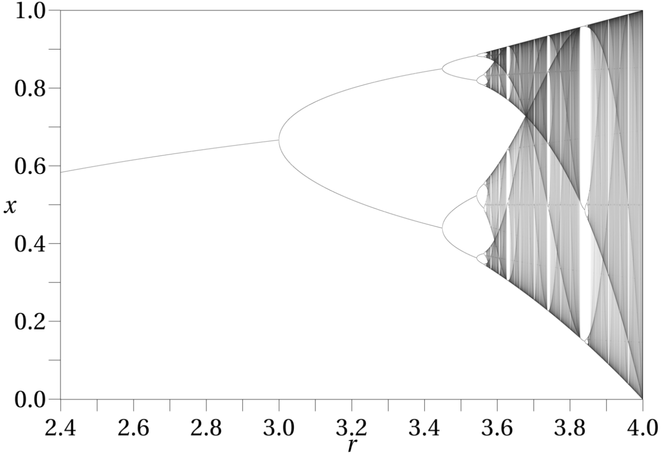
Since p_{t+1} grows, it *cannot* be concluded that $p_t \to 0$.





Bifurcation diagram of the logistic map $P_{t+1} = rP_t(1 - P_t)$

The following diagram show the long-term values (i.e., equilibria, fixed points, or periodic orbits) as a function of bifurcation parameter, r.



What do you notice?