Predator-prey models

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

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Introduction

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Simple predator-prey model

$$\begin{cases} \Delta P = rP(1 - P/M) - sPQ \\ \Delta Q = -uQ + vPQ \end{cases} \qquad r, s, u, v, K > 0,\ u < 1$$

Predator-prey model

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- phase plots: Q_t vs. P_t.

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Consider the following predator-prey model:

$$\begin{cases} P_{t+1} = P_t(1 + 1.3(1 - P_t)) - .5P_tQ_t \\ Q_{t+1} = .3Q_t + 1.6P_tQ_t \end{cases}$$

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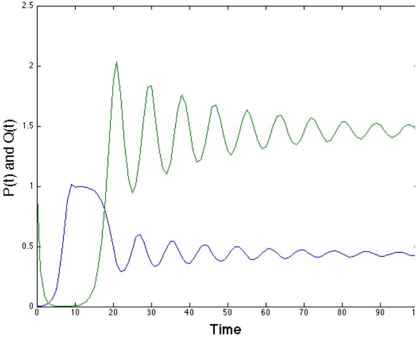
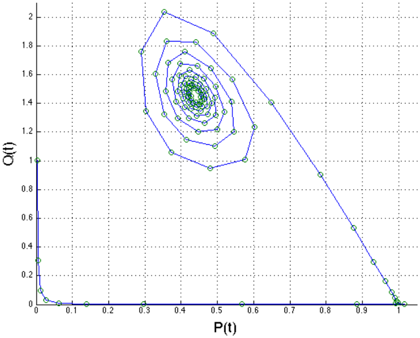
Solutions can be graphed using a *time plot* (left) or a *phase plot* (right):





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Thus, there are three equilibria:

$$(P^*, Q^*) = (0, 0), \quad (1, 0), \quad (.4375, 1.4625).$$

Equilibria and nullclines

For the general predator-prey model:

$$\begin{cases} P_{t+1} = P_t(1 + r(1 - P_t/M)) - sP_tQ_t \\ Q_{t+1} = (1-u)Q_t + vP_tQ_t \end{cases} \qquad r, s, u, v, K > 0, \quad u < 1$$

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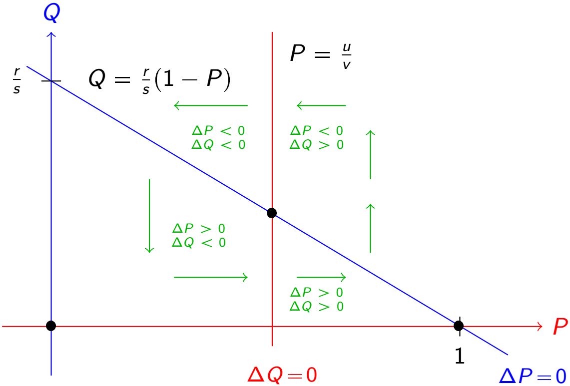
A *nullcline* is a line on which either $\Delta P = 0$ or $\Delta Q = 0$. In our example:

$$P = 0, \qquad Q = \frac{r}{s}(1 - P), \qquad Q = 0, \qquad P = \frac{u}{v}.$$

Nullclines

We can plot the nullclines on the PQ-plane to help visualize the dynamics.



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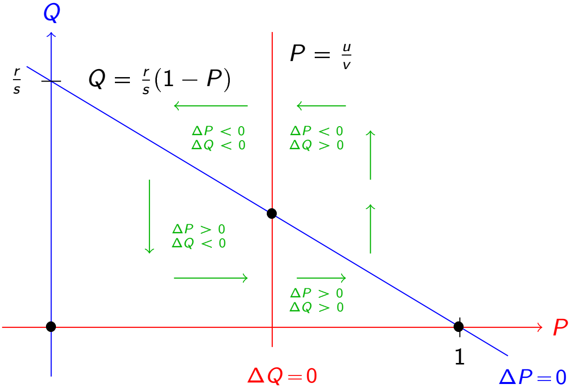

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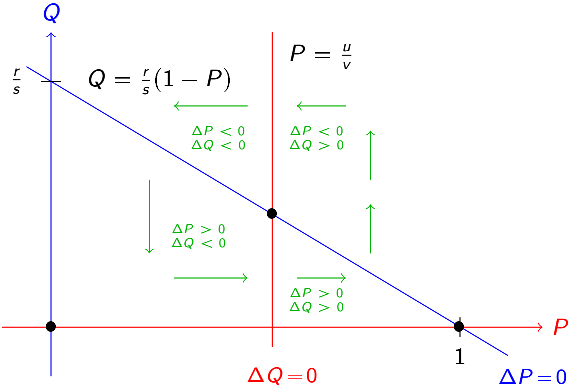

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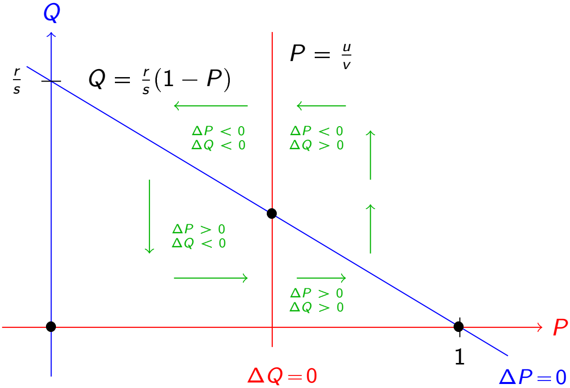

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Do you see how we determine the direction of the green arrows? Can we tell whether it spirals inward or outward?

Nullclines

Remark

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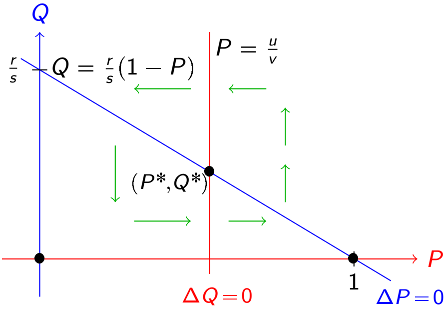

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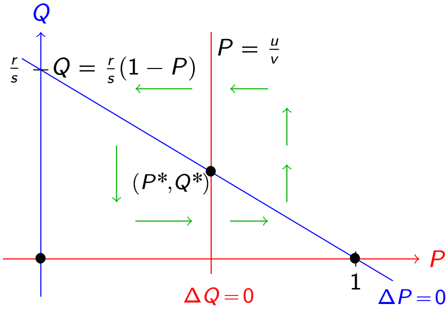

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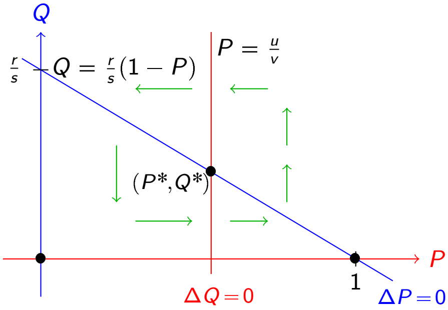

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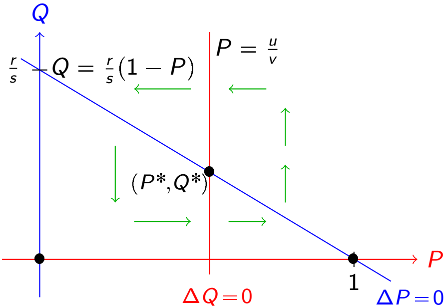

Suppose the predator was an insect and the prey was an agricultural crop.

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Unfortunately, this won't work: P^* is unchanged, but Q^* increases. (Why?)

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and simplifying yields

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For small perturbations (p_t, q_t), we can neglect the nonlinear terms (e.g., p_t^2, q_t^2, and p_tq_t) which are ≈ 0, leaving a linear system $\mathbf{p}_{t+1} \approx \mathbf{A}\mathbf{p}_t$.

Linearization (cont.)

Thus, given a small perturbation (p_t, q_t) at time t, it can be described at time $t+1$ by a linear equation $\mathbf{p}_{t+1} \approx \mathbf{A}\mathbf{p}_t$:

$$\begin{bmatrix} p_{t+1} \\ q_{t+1} \end{bmatrix} \approx \begin{bmatrix} .43125 & -.21875 \\ 2.34 & 1 \end{bmatrix} \begin{bmatrix} p_t \\ q_t \end{bmatrix}.$$

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