

# Predator-prey models

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## Introduction

Consider a population of two species, e.g., foxes (“predator”) and rabbits (“prey”).

- $P_t$  = size of prey.
- $Q_t$  = size of predator.

The change in population size of each is a function of *both* population sizes:

$$\Delta P = F(P, Q), \quad \Delta Q = G(P, Q).$$

### Question

What would happen if the predator or the prey disappeared?

- Prey, *without* predators:  $\Delta P = r(P(1 - P_t/M))$ .
- Predators, *without* prey:  $\Delta Q = -uQ$ , where  $u \in (0, 1)$  is per-capita death rate.

### Simple predator-prey model

$$\begin{cases} \Delta P = rP(1 - P/M) - sPQ \\ \Delta Q = -uQ + vPQ \end{cases} \quad r, s, u, v, K > 0, u < 1$$

## Predator-prey model

### Alternate form

$$\begin{cases} P_{t+1} = P_t(1 + r(1 - P_t/M)) - sP_tQ_t \\ Q_{t+1} = (1 - u)Q_t + vP_tQ_t \end{cases} \quad r, s, u, v, K > 0, \quad u < 1$$

The  $-sPQ$  and  $vPQ$  are called *mass-action terms*. Roughly speaking:

- $-sPQ$  describes a *negative* effect of the predator-prey interaction on the prey,
- $vPQ$  describes a *positive* effect of the predator-prey interaction on the predator.

Qualitatively, larger values of  $s$  and  $v$  indicate stronger predator-prey interaction.

We can plot the solutions of these equations several ways:

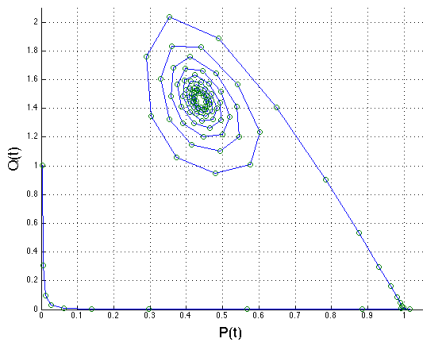
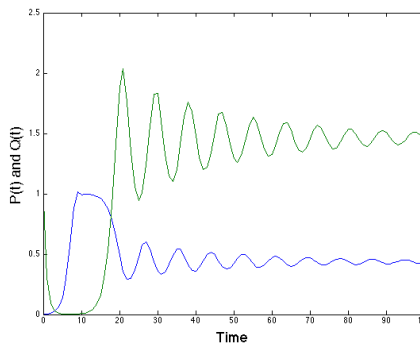
- **time plots:**  $P_t$  vs.  $t$ , and  $Q_t$  vs.  $t$
- **phase plots:**  $Q_t$  vs.  $P_t$ .

## Time plots and phase plots

Consider the following predator-prey model:

$$\begin{cases} P_{t+1} = P_t(1 + 1.3(1 - P_t)) - .5P_tQ_t \\ Q_{t+1} = .3Q_t + 1.6P_tQ_t \end{cases}$$

Solutions can be graphed using a *time plot* (left) or a *phase plot* (right):



## Equilibria

To find steady-state population(s), we set  $P_t = P_{t+1} = P^*$  and  $Q_t = Q_{t+1} = Q^*$ .

$$\begin{cases} P_{t+1} = P_t(1 + 1.3(1 - P_t)) - .5P_tQ_t \\ Q_{t+1} = .3Q_t + 1.6P_tQ_t \end{cases} \rightsquigarrow \begin{cases} P^* = P^*(1 + 1.3(1 - P^*)) - .5P^*Q^* \\ Q^* = .3Q^* + 1.6P^*Q^* \end{cases}$$

Via simple algebra, this reduces to the following system

$$\begin{cases} 0 = P^*(1.3 - 1.3P^* - .5Q^*) \\ 0 = Q^*(-.7 + 1.6P^*) \end{cases}$$

If  $Q^* = 0$ , then  $P^* = 0$  or  $P^* = 1$ .

Alternatively,  $P^* = .4375$ , which would force  $Q^* = 1.4625$ .

Thus, there are three equilibria:

$$(P^*, Q^*) = (0, 0), (1, 0), (.4375, 1.4625).$$

## Equilibria and nullclines

For the general predator-prey model:

$$\begin{cases} P_{t+1} = P_t(1 + r(1 - P_t/M)) - sP_tQ_t \\ Q_{t+1} = (1 - u)Q_t + vP_tQ_t \end{cases} \quad r, s, u, v, K > 0, \quad u < 1$$

the equilibrium equations (set  $P_t = P_{t+1} = P^*$  and  $Q_t = Q_{t+1} = Q^*$ ) are

$$\begin{cases} 0 = P^*(r(1 - P^*) - sQ^*) \\ 0 = Q^*(-u + vP^*). \end{cases}$$

For Equation 2 to be satisfied,  $Q^* = 0$  or  $-u + vP^* = 0$ .

Furthermore, Equation 1 is satisfied if  $P^* = 0$  or  $r(1 - P^*) - sQ^* = 0$ .

By simple algebra, we get three equilibria:

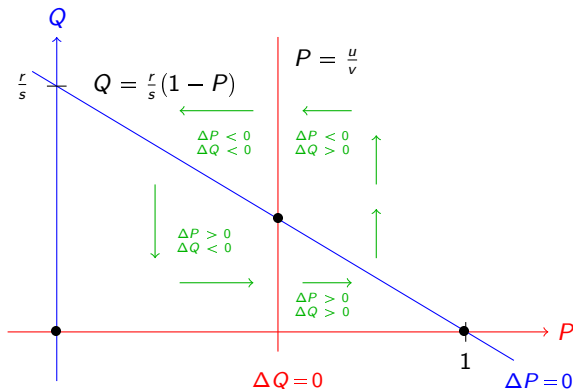
$$(P^*, Q^*) = (0, 0), \quad (1, 0), \quad \left(\frac{u}{v}, \frac{r}{s}\left(1 - \frac{u}{v}\right)\right).$$

A *nullcline* is a line on which either  $\Delta P = 0$  or  $\Delta Q = 0$ . In our example:

$$P = 0, \quad Q = \frac{r}{s}(1 - P), \quad Q = 0, \quad P = \frac{u}{v}.$$

## Nullclines

We can plot the nullclines on the  $PQ$ -plane to help visualize the dynamics.



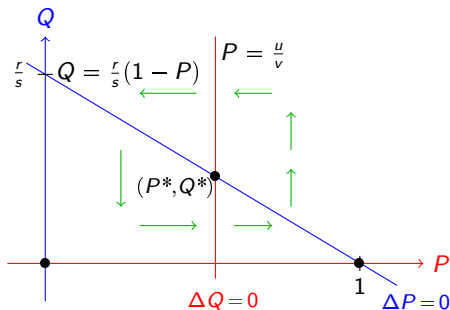
- $\Delta P > 0$  occurs *below*  $Q = \frac{r}{s}(1 - P)$ .
- $\Delta Q > 0$  occurs *to the right of*  $P = \frac{u}{v}$ .

Do you see how we determine the direction of the green arrows? Can we tell whether it spirals inward or outward?

## Nullclines

### Remark

Changing  $r$  or  $s$  doesn't affect the  $Q$ -nullcline.



Suppose the **predator** was an **insect** and the **prey** was an **agricultural crop**.

One might want to introduce a new crop variety with higher  $r$ , to try to “outgrow” the predator.

Unfortunately, this won't work:  $P^*$  is unchanged, but  $Q^*$  increases. (Why?)



## Linearization

Suppose  $(P^*, Q^*)$  is a fixed point whose stability we wish to understand.

We can plug the following “perturbation” back into the original system:

$$P_t = P^* + p_t, \quad P_{t+1} = P^* + p_{t+1}, \quad Q_t = Q^* + q_t, \quad Q_{t+1} = Q^* + q_{t+1}.$$

Consider the fixed point  $(P^*, Q^*) = (.4375, 1.4625)$  of our previous example.

Plugging

$$P_t = .4375 + p_t, \quad P_{t+1} = .4375 + p_{t+1}, \quad Q_t = 1.4625 + q_t, \quad Q_{t+1} = 1.4625 + q_{t+1}.$$

$$\text{into} \quad \begin{cases} P_{t+1} = P_t(1 + 1.3(1 - P_t)) - .5P_tQ_t \\ Q_{t+1} = .3Q_t + 1.6P_tQ_t \end{cases}$$

and simplifying yields

$$\begin{cases} p_{t+1} = .43125p_t - .21875q_t - 1.3p_t^2 - .5p_tq_t \\ q_{t+1} = 2.34p_t + q_t + 1.6p_tq_t \end{cases}$$

For small perturbations  $(p_t, q_t)$ , we can neglect the nonlinear terms (e.g.,  $p_t^2$ ,  $q_t^2$ , and  $p_tq_t$ ) which are  $\approx 0$ , leaving a linear system  $\mathbf{p}_{t+1} \approx \mathbf{A}\mathbf{p}_t$ .

## Linearization (cont.)

Thus, given a small perturbation  $(p_t, q_t)$  at time  $t$ , it can be described at time  $t + 1$  by a linear equation  $\mathbf{p}_{t+1} \approx \mathbf{A}\mathbf{p}_t$ :

$$\begin{bmatrix} p_{t+1} \\ q_{t+1} \end{bmatrix} \approx \begin{bmatrix} .43125 & -.21875 \\ 2.34 & 1 \end{bmatrix} \begin{bmatrix} p_t \\ q_t \end{bmatrix}.$$

The eigenvalues of  $\mathbf{A}$  are  $\lambda = .7156 \pm .6565i$ , which have norm

$$|\lambda| = \sqrt{(.7156)^2 + (.6565)^2} = .9711 < 1.$$

Thus, this perturbation from the steady-state is **shrinking**. The population will spiral back into the steady-state  $(P^*, Q^*) = (.4375, 1.4625)$ .

### Types of equilibrium points

- $|\lambda_1| < 1, |\lambda_2| < 1$ , stable
- $|\lambda_1| > 1, |\lambda_2| > 1$ , unstable
- $|\lambda_1| < 1, |\lambda_2| > 1$ , saddle

## Other interaction models

- **Competition:** 2 species fill the same niche in an environment.

$$\begin{cases} \Delta P = rP(1 - (P + Q)/K) \\ \Delta Q = rQ(1 - (P + Q)/K) \end{cases}$$

*Question:* Does one species “win”? Or can they co-exist?

- **Competition with predator/prey:** 
$$\begin{cases} \Delta P = rP(1 - (P + Q)/K) - sPQ \\ \Delta Q = rQ(1 - (P + Q)/K) \pm vPQ \end{cases}$$

- **Mutualism:** e.g.,  $P$  = sharks,  $Q$  = feeder fish. 
$$\begin{cases} \Delta P = rP(1 - P/K) + sPQ \\ \Delta Q = -uQ + vPQ \end{cases}$$

- **Immune system vs. infective agent:**

$$\begin{array}{l} P : \text{immune cells} \\ Q : \text{level of infection} \end{array} \quad \begin{cases} \Delta P = rQ - sPQ \\ \Delta Q = uQ - vPQ \end{cases}$$

- $-sPQ$ : negative effect on immune system from fighting
- $-sPQ$ : limited effect on immune system from fighting
- $rQ$ : immune response is proportional to infection level