Linear models of structured populations

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$E_t = \#$ eggs at time t

$L_t = \#$ larve at time t

$A_t = \#$ adults at time t

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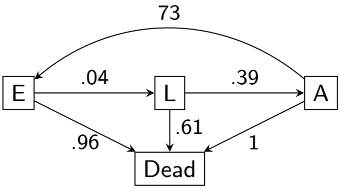

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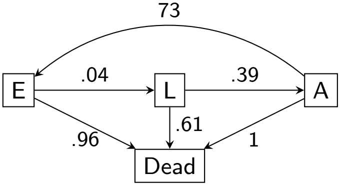

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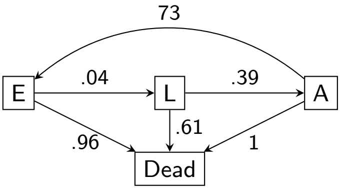

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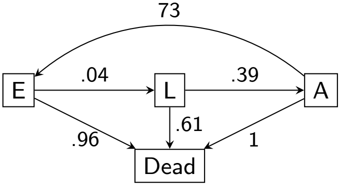

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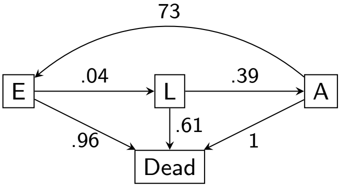

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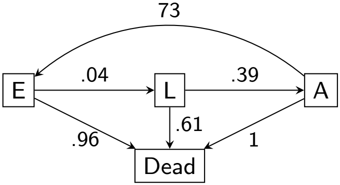

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Thus, this is just exponential growth. But what if instead of dying, 65% of adults survive another day?

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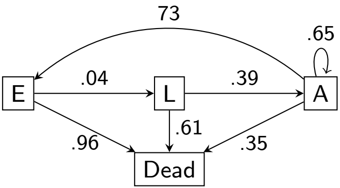

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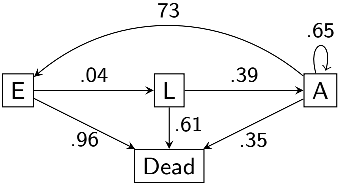

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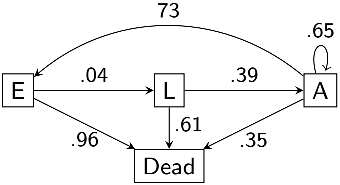

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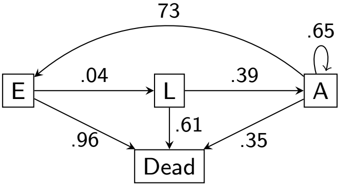

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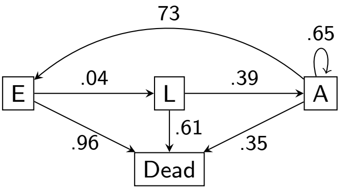

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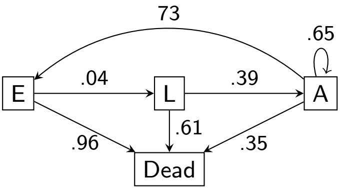

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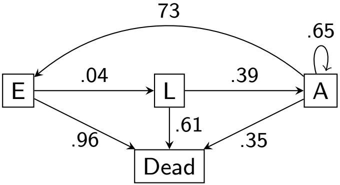

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- How much effect does changing the initial conditions have?

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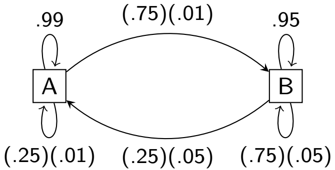

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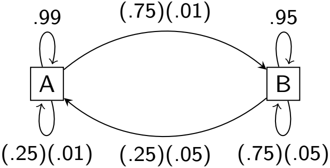

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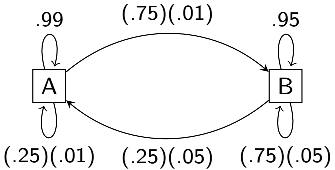

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$$\mathbf{c} = P^{-1}\mathbf{x}_0 = -\frac{1}{8} \begin{bmatrix} -1 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 10 \\ 990 \end{bmatrix} = \begin{bmatrix} 125 \\ -615 \end{bmatrix}$$

An example, revisted

Let us revisit our "tree example", where $P = \begin{bmatrix} .9925 & .0125 \\ .0075 & .9875 \end{bmatrix}$.

The eigenvalues and eigenvectors of P are

$$\lambda_1 = 1, \ \ \mathbf{v}_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \qquad \lambda_2 = .98, \ \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Consider the initial condition $\mathbf{x}_0 = \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} 10 \\ 990 \end{bmatrix}$.

First step

Write $\mathbf{x}_0 = c_1 \begin{bmatrix} 5 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, i.e., solve $P\mathbf{c} = \mathbf{x}_0$:

$$\begin{bmatrix} 5 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 990 \end{bmatrix}.$$

$$\mathbf{c} = P^{-1}\mathbf{x}_0 = -\frac{1}{8} \begin{bmatrix} -1 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 10 \\ 990 \end{bmatrix} = \begin{bmatrix} 125 \\ -615 \end{bmatrix}$$

Thus, our initial vector is $\mathbf{x}_0 = \begin{bmatrix} 10 \\ 990 \end{bmatrix} = 125 \begin{bmatrix} 5 \\ 3 \end{bmatrix} - 615 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

An example (cont.)

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Solving for $\mathbf{x}_t$

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Once we have written $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, the solution $\mathbf{x}_t$ is simply

$$\mathbf{x}_t = P^t\mathbf{x}_0$$

An example (cont.)

Solving for $\mathbf{x}_t$

Once we have written $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, the solution $\mathbf{x}_t$ is simply

$$\mathbf{x}_t = P^t\mathbf{x}_0 = c_1\lambda_1^t\mathbf{v}_1 + c_2\lambda_2^t\mathbf{v}_2 \,.$$

An example (cont.)

Solving for $\mathbf{x}_t$

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In our example, $\mathbf{x}_0 = 125\begin{bmatrix}5\\3\end{bmatrix} - 615\begin{bmatrix}1\\-1\end{bmatrix}$,

An example (cont.)

Solving for $\mathbf{x}_t$

Once we have written $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, the solution $\mathbf{x}_t$ is simply

$$\mathbf{x}_t = P^t\mathbf{x}_0 = c_1\lambda_1^t\mathbf{v}_1 + c_2\lambda_2^t\mathbf{v}_2\,.$$

In our example, $\mathbf{x}_0 = 125\begin{bmatrix}5\\3\end{bmatrix} - 615\begin{bmatrix}1\\-1\end{bmatrix}$, and so

$$\mathbf{x}_t = 125(1)^t\begin{bmatrix}5\\3\end{bmatrix} - 615(.98)^t\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}625-(615)(.98)^t\\375+(615)(.98)^t\end{bmatrix}.$$

An example (cont.)

Solving for $\mathbf{x}_t$

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The long-term behavior of this system is

$$\lim_{t\to\infty}\mathbf{x}_t = 125\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}625\\375\end{bmatrix}\,.$$

An example (cont.)

Solving for $\mathbf{x}_t$

Once we have written $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, the solution $\mathbf{x}_t$ is simply

$$\mathbf{x}_t = P^t\mathbf{x}_0 = c_1\lambda_1^t\mathbf{v}_1 + c_2\lambda_2^t\mathbf{v}_2\,.$$

In our example, $\mathbf{x}_0 = 125\begin{bmatrix}5\\3\end{bmatrix} - 615\begin{bmatrix}1\\-1\end{bmatrix}$, and so

$$\mathbf{x}_t = 125(1)^t\begin{bmatrix}5\\3\end{bmatrix} - 615(.98)^t\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}625-(615)(.98)^t\\375+(615)(.98)^t\end{bmatrix}.$$

The long-term behavior of this system is

$$\lim_{t\to\infty}\mathbf{x}_t = 125\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}625\\375\end{bmatrix}\,.$$

Notice that this does *not* depend on $\mathbf{x}_0$!