1. Let $T\colon X \to U$ be a linear map. Prove the following:
   (a) The image of a subspace of X is a subspace of U.
   (b) The inverse image of a subspace of U is a subspace of X.

2. Prove Theorem 3.3 in Lax:
   (a) The composite of linear mappings is also a linear mapping.
   (b) Composition is distributive with respect to the addition of linear maps, that is,

   $$(R+S)\circ T = R\circ T + S\circ T$$

   and

   $$S\circ(T+P) = S\circ T + S\circ P\,,$$

   where $R, S\colon U \to V$ and $P, T\colon X \to U$.

3. Show that whenever meaningful,
   (a) $(ST)' = T'S'$
   (b) $(T+R)' = T' + R'$
   (c) $(T^{-1})' = (T')^{-1}$.

   Here, S' denotes the transpose of S. Carefully describe what you mean by "whenever meaningful" in each case.

4. Give a direct algebraic proof of $N_{T'}^{\perp} = (R_T^{\perp})^{\perp}$. (You may use the fact that $N_{T'} = R_T^{\perp}$, but don't simply take the annihilator of both sides of this equation.)

5. Suppose $T\colon X \to X$ is a linear map of rank 1.
   (a) Show that there exists $c \in K$ such that $T^2 = cT$.
   (b) Show that if $c \neq 1$, then $I - T$ has an inverse.

6. Suppose that $S, T\colon X \to X$ are linear maps.
   (a) Show that $\operatorname{rank}(S+T) \leq \operatorname{rank}(S) + \operatorname{rank}(T)$.
   (b) Show that $\operatorname{rank}(ST) \leq \operatorname{rank}(S)$.
   (c) Show that $\dim(N_{ST}) \leq \dim N_S + \dim N_T$.

   For each of these, give an explicit example showing how equality need not hold.