Throughout, X is assumed to be a vector space of dimension $n < \infty$.

1. Let $A, B \colon X \to X$ be linear maps.

   (a) Show that if A is invertible and similar to B, then B is also invertible, and B^{-1} is similar to A^{-1}.

   (b) Show that if either A or B is invertible, then AB and BA are similar.

2. Let $T : X \to X$ be linear, with $\dim X = n$.

   (a) Prove that if $T^2 = T$, then $X = R_T \oplus N_T$.

   (b) Show by example that if $T^2 \neq T$, then $X = R_T \oplus N_T$ need not hold.

   (c) Prove that $N_{T^n} = N_{T^{n+1}}$ and $R_{T^n} = R_{T^{n+1}}$.

   (d) Prove that $X = R_{T^n} \oplus N_{T^n}$.

   (e) Show there exists a linear map $S : X \to X$ such that $ST = TS$ and $ST^{n+1} = T^n$.

3. Let X and U be vector spaces, and suppose that Y is a subspace of X. Let $Q \colon X \to X/Y$ be the canonical quotient map sending $x \stackrel{Q}{\longmapsto} \{x\}$, and let $T \colon X \to U$ be a linear map. Give necessary and sufficient conditions for the existence of a unique linear map $S \colon X/Y \to U$ such that $T = S \circ Q$. When this happens, the map T is said to *factor through* the quotient space, as shown by the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "Q"'] & & U \\ & X/Y \arrow[ur, dashed, "S"'] & \end{tikzcd}$$

   Prove all of your claims.

4. Let $\mathcal{P}_n$ be the vector space of all polynomials over $\mathbb{R}$ of degree less than n.

   (a) Show that the map $T \colon \mathcal{P}_3 \to \mathcal{P}_4$ given by

   $$T(p(x)) = 6 \int_1^x p(t)\, dt$$

   is linear. Indicate whether it is 1–1 or onto.

   (b) Let $\mathcal{B}_3 = \{1, x, x^2\}$ be a basis for $\mathcal{P}_3$ and let $\mathcal{B}_4 = \{1, x, x^2, x^3\}$ be a basis for $\mathcal{P}_4$. Find the matrix representation of T with respect to these bases.

5. Let $T\colon X \to U$, with $\dim X = n$ and $\dim U = m$. Show that there exist bases $\mathcal{B}$ for X and $\mathcal{B}'$ for U such that the matrix of T in block form is

$$M = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$

where I_k is the $k \times k$ identity matrix, and the other blocks are either empty or contain all zeros.

6. Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ with matrix representation $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ -3 & 0 & 3 \end{bmatrix}$ with respect to the standard basis. What is the matrix representation of T with respect to the basis $\{(1,-1,0),(0,1,-1),(1,0,1)\}$?