*Read*: Lax, Chapter 5, pages 44–57.

1. Let $S_n$ denote the set of all permutations of $\{1, \dots, n\}$.
   (a) Prove that $\operatorname{sgn}(\pi_1 \circ \pi_2) = \operatorname{sgn}(\pi_1)\operatorname{sgn}(\pi_2)$.
   (b) Let $\pi \in S_n$, and suppose that $\pi = \tau_k \circ \cdots \circ \tau_1 = \sigma_\ell \circ \cdots \circ \sigma_1$, where $\tau_i, \sigma_j \in S_n$ are transpositions. Prove that $k \equiv \ell \bmod 2$.

2. Let $X$ be an $n$-dimensional vector space over a field $K$.
   (a) Prove that if the characteristic of $K$ is not 2, then every skew-symmetric form is alternating.
   (b) Give an example of a non-alternating skew-symmetric form.
   (c) Give an example of a non-zero alternating $k$-linear form ($k < n$) such that $f(x_1, \dots, x_k) = 0$ for some set of linearly independent vectors $x_1, \dots, x_k$.

3. Let $X$ be a 2-dimensional vector space over $\mathbb{C}$, and let $f \colon X \times X \to \mathbb{C}$ be an alternating, bilinear form. If $\{x_1, x_2\}$ is a basis of $X$, determine a formula for $f(u, v)$ in terms of $f(x_1, x_2)$, and the coefficients used to express $u$ and $v$ with this basis. [Pun intented!]

4. Let $X$ be an $n$-dimensional vector space over $\mathbb{R}$, and let $f$ be a non-degenerate symmetric bilinear form. That is, it has the additional property that for all nonzero $x \in X$, there is some $y \in X$ for which $f(x, y) \neq 0$.
   (a) Prove that the map $L \colon X \to X'$ given by $L \colon x \mapsto f(x, -)$ is an isomorphism.
   (b) Show that, given any basis $x_1, \dots, x_n$ for $X$, there exists a basis $y_1, \dots, y_n$ such that $f(x_i, y_j) = \delta_{ij}$.
   (c) Conversely, prove that if $\mathcal{B}_X = \{x_1, \dots, x_n\}$ and $\mathcal{B}_Y = \{y_1, \dots, y_n\}$ are sets of vectors in $X$ with $f(x_i, y_j) = \delta_{ij}$, then $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for $X$.

5. Let $X$ be an $n$-dimensional vector space over $\mathbb{R}$, and let $f$ be a non-degenerate symmetric bilinear form.
   (a) Show that there exists $x_1 \in X$ with $f(x_1, x_1) \neq 0$.
   (b) Any fixed $x_1 \in X$ for which $f(x_1, x_1) \neq 0$ induces a *linear* map $T = f(x_1, -)$. Show that the nullspace $N_T$ has dimension $n - 1$.
   (c) Let $Z_1 = N_T$. Show that the restriction of $f$ to $Z_1 \times Z_1$ is again non-degenerate.
   (d) Prove that $X$ has a basis $\{x_1, \dots, x_n\}$ such that $f(x_i, x_i) \neq 0$ for all $i$, and $f(x_i, x_j) = 0$ whenever $i \neq j$.
   (e) Give an example of a vector space $X$ ($2 \leq \dim X < \infty$) with basis $\mathcal{B}$ and a non-degenerate symmetric bilinear form $f$ for which $f(x, x) = 0$ for all $x \in \mathcal{B}$.

6. Let $A = (c_1, \dots, c_n)$ be an $n \times n$ matrix ($c_i$ is a column vector), and let $B$ be the matrix obtained from $A$ by adding $k$ times the $i^{\text{th}}$ column of $A$ to the $j^{\text{th}}$ column of $A$, for $i \neq j$. Prove that $\det A = \det B$.