Read: Lax, Chapter 6, pages 58–69.

1. Prove the following properties of the trace function:
   (a) $\operatorname{tr} AB = \operatorname{tr} BA$ for all $m \times n$ matrices A and $n \times m$ matrices B.
   (b) $\operatorname{tr} AA^T = \sum a_{ij}^2$ for all $n \times n$ matrices A. The quantity is the square of the *Hilbert-Schmidt norm* of A.

2. (a) Show that if A and B are similar, then A and B have the same eigenvalues.
   (b) Is the converse of Part (a) true? Prove or disprove.

3. Let A be a 2×2 matrix over $\mathbb{R}$ satisfying $A^T = A$. Prove that A has 2 linearly independent eigenvectors in $\mathbb{R}^2$.

4. Consider the following matrices:
$$A = \begin{bmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -4 & 85 \\ 1 & 4 & -30 \\ 0 & 0 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}.$$
   A straightforward calculation shows that the characteristic polynomials are
$$p_A(s) = p_B(s) = p_C(s) = (s-2)^2(s-3)\,.$$
   (a) Determine the eigenvectors and the minimal polynomials of each matrix
   (b) Find a basis $\{v_1, v_2, v_3\}$ for $\mathbb{R}^3$ where $Bv_1 = 3v_1$, $Bv_2 = 2v_2$, and $(B-2I)v_3 = v_2$. Write the matrix of this linear map with respect to this new basis.
   (c) Repeat the previous step for the matrix C.

5. Let A_θ be a 3×3 matrix representing a rotation of $\mathbb{R}^3$ through an angle θ about the y-axis.
   (a) Find the eigenvalues for A_θ over $\mathbb{C}$.
   (b) Determine necessary and sufficient conditions on θ in order for A_θ to contain three linearly independent eigenvectors in $\mathbb{R}^3$. Justify your claim and interpret it geometrically.

6. Let A be an invertible $n \times n$ matrix. Prove that A^{-1} can be written as a polynomial in degree at most $n-1$. That is, prove that there are scalars c_i such that
$$A^{-1} = c_{n-1}A^{n-1} + c_{n-2}A^{n-2} + \cdots c_1 A + c_0 I\,.$$

7. Let A be an $n \times n$ matrix over $\mathbb{C}$ with distinct eigenvalues $\lambda_1, \dots, \lambda_n$. For a vector $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, define the *norm* of z by
$$||z|| = \Big(\sum_{i=1}^{n} |z_i|\Big)^{1/2}.$$
   (a) Prove that if $|\lambda_i| < 1$ for all i, then $||A^N z|| \to 0$ as $N \to \infty$ for all $z \in \mathbb{C}^n$.
   (b) Prove that if $|\lambda_i| > 1$ for all i, then $||A^N z|| \to \infty$ as $N \to \infty$ for all nonzero $z \in \mathbb{C}^n$.