Read: Lax, Chapter 6, pages 69–76.

1. Do the following for the matrix A below, and then repeat it for B:

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 0 & -6 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & -4 \\ 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   (a) Find the characteristic polynomial and all (genuine) eigenvectors.
   (b) For each eigenvalue λ, compute $\dim N_{(A-\lambda I)^j}$ for $j = 1, 2, 3, \dots$.
   (c) Find a basis of $\mathbb{C}^4$ consisting of generalized eigenvectors.
   (d) Factor the matrix as a product PJP^{-1}, where the columns of P are the generalized eigenvectors and J is the matrix of the linear map with respect to this new basis.

2. Let A be an $n \times n$ matrix over $\mathbb{C}$ with an eigenvalue λ of index $m \geq 2$ and corrresponding eigenvector v_1. Let v_2 be a generalized eigenvector satisfying $(A - \lambda I)v_2 = v_1$.

   (a) Prove that for any natural number N,
   $$A^N v_2 = \lambda^N v_2 + N\lambda^{N-1} v_1 .$$
   (b) Prove that for any polynomial $q(t) \in \mathbb{C}[t]$,
   $$q(A)v_2 = q(\lambda)v_2 + q'(\lambda)v_1 ,$$
   where $q'(t)$ is the derivative of q.
   (c) Conjecture a formula for $q(A)v_m$, where $v_1, \dots, v_m$ are generalized eigenvectors of A with $(A - \lambda I)v_k = v_{k-1}$ (and say $v_0 = 0$, for convenience).

3. Let λ be an eigenvalue of A, and let N_i be the nullspace of $(A - \lambda I)^i$. Prove that $A - \lambda I$ extends to a well-defined map $N_{i+1}/N_i \longrightarrow N_i/N_{i-1}$, and that this mapping is 1–1.

4. Let X be an n-dimensional vector space, and $A\colon X \to X$ a linear map with distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Let $v_1, \dots, v_n$ be the corresponding eigenvectors of A, and let $\ell_1, \dots, \ell_n$ be the corresponding eigenvectors of the transpose $A'\colon X' \to X'$.

   (a) Prove that $(\ell_i, v_i) \neq 0$ for $i = 1, \dots, n$.
   (b) Show that if $x = a_1 v_1 + \cdots + a_n v_n$, then $a_i = (\ell_i, x)/(\ell_i, v_i)$.
   (c) Is $\ell_1, \dots, \ell_n$ necessarily the dual basis of $v_1, \dots, v_n$? Why or why not?