

Read: Lax, Appendix 15, pages 363–366.

1. Let A be a 7×7 matrix over \mathbb{C} with minimal polynomial $p(s) = (s - 1)^3(s - 2)^2$.
 - (a) List all possible Jordan canonical forms of A and justify the completeness of your list.
 - (b) For each matrix from Part (a), find the rank of $(A - I)^k$ and $(A - 2I)^k$, for $k \in \mathbb{N}$.

2. Let A be an $n \times n$ matrix over \mathbb{C} . The matrix A is *nilpotent* if $A^k = 0$ for some $k \in \mathbb{N}$.
 - (a) Prove that if A is nilpotent, then $A^n = 0$.
 - (b) Prove that if A is nilpotent, then there is some $r \in \mathbb{N}$ and positive integers $k_1 \geq \dots \geq k_r$ with $k_1 + \dots + k_r = n$ that determine A up to similarity.
 - (c) Suppose A and B are 6×6 nilpotent matrices with the same minimal polynomial and $\dim N_A = \dim N_B$. Prove that A and B are similar. Show by example that this can fail for 7×7 matrices.

3. Let A and B be $n \times n$ matrices over \mathbb{C} . The matrix A is *idempotent* if $A^2 = A$.
 - (a) Prove that if $A^k = A$ for some integer $k > 1$, then A is diagonalizable.
 - (b) Prove that idempotent matrices are similar if and only if they have the same trace.
 - (c) Prove that if A and B are idempotent and $B = UAV$ holds for some invertible maps $U, V: X \rightarrow X$, then A and B are similar.

4. Let X be an n -dimensional vector space over \mathbb{C} , and let $A, B: X \rightarrow X$ be linear maps.
 - (a) Prove that if $AB = BA$, then for any eigenvector v of A with eigenvalue λ , the vector Bv is an eigenvector of A for λ .
 - (b) Show that if $\{A_1, A_2, \dots \mid A_i: X \rightarrow X\}$ is a (possibly infinite) set of pairwise commuting maps, then there is a nonzero $x \in X$ that is an eigenvector of every A_i .
 - (c) Suppose that A and B are both diagonalizable. Prove that $AB = BA$ if and only if they are *simultaneously diagonalizable*, i.e., there exists an invertible $n \times n$ -matrix P such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.

5. Let $X \subset \mathbb{R}[x, y]$ be the space of polynomials in x, y of total degree $\leq n$. Find a basis of X , and find the Jordan canonical form of the linear map

$$A: X \longrightarrow X, \quad f \longmapsto f + \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

[*Hint:* First find all genuine eigenvectors.]