Read: Lax, Chapter 7, pages 77–100.

1. This problem is about *rational canonical form.* Consider the following matrix:

$$M = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

   (a) Show that the minimal polynomial of M is

$$f(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \,.$$

   (b) Let X be a 4 dimensional vector space over $\mathbb{R}$ with basis $\{x_1, x_2, x_3, x_4\}$ and let $T : X \to X$ be a linear map such that

$$T(x_1) = x_2, \quad T(x_2) = x_3, \quad T(x_3) = x_4, \quad T(x_4) = -x_1 - 4x_2 - 6x_3 - 4x_4.$$

   Is T diagonalizable over $\mathbb{C}$? Why or why not?

2. Prove that $||x|| = \max\big\{(x, y) : y \in K^n \text{ with } ||y|| = 1\big\}$.

3. Let f and g be continuous functions on the interval $[0, 1]$. Prove the following inequalities.

   (a) $\displaystyle\left(\int_0^1 f(t)g(t)\,dt\right)^2 \le \int_0^1 f(t)^2\,dt \int_0^1 g(t)^2\,dt$

   (b) $\displaystyle\left(\int_0^1 (f(t)+g(t))^2\,dt\right)^{1/2} \le \left(\int_0^1 f(t)^2\,dt\right)^{1/2} + \left(\int_0^1 g(t)^2 dt\right)^{1/2}.$

4. Use the Gram-Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $y_1 = (1, 2, 1, 1)$, $y_2 = (1, -1, 0, 2)$ and $y_3 = (2, 0, 1, 1)$.

5. Let X be the vector space of all continuous real-valued functions on $[0, 1]$. Define an inner product on X by

$$(f, g) = \int_0^1 f(t)g(t)\,dt\,.$$

   Let Y be the subspace of X spanned by f_0, f_1, f_2, f_3, where $f_k(x) = x^k$. Find an orthonormal basis for Y.