Read: Lax, Chapter 7, pages 89–100.

1. Let Y be a subspace of a Euclidean space X, and $P_Y \colon X \to X$ the orthogonal projection onto Y. Prove that $P_Y^* = P_Y$.

2. Let X be a finite-dimensional real Euclidean space. We say that a sequence $\{A_n\}$ of linear maps converges to a limit A if $\lim_{n\to\infty} ||A_n - A|| = 0$.

   (a) Show that $\{A_n\}$ converges to A if and only if for all $x \in X$, $A_n x$ converges to Ax.

   (b) Show by example that this fails if $\dim X = \infty$.

3. Let $A \colon X \to U$ be a linear map between Euclidean spaces, and let $A^* \colon U \to X$ denote the adjoint map. The map A has a *left inverse* if there is a linear map $L \colon U \to X$ such that $LA = I_X$, the identity on X. It has a *right inverse* if there is a linear map $R \colon U \to X$ such that $AR = I_U$ is the identity on U.

   (a) Prove that $R_{A^*}^{\perp} = N_A$.

   (b) Prove that A maps R_{A^*} bijectively onto R_A.

   (c) Show that if A has a left inverse, then $Ax = u$ has *at most* one solution. Give a condition on u that completely characterizes when there is a solution.

   (d) Show that if A has a right inverse, then $Ax = u$ has *at least* one solution. If $Ax_p = u$ for some particular $x_p \in X$, then describe all solutions for x in this case. What condition ensures that there will be *only* one solution?

   (e) What are the possibilities for the rank of A if it has a left inverse? What if it has a right inverse?

4. Let X be the space of continuous complex-valued functions on $[-1, 1]$ and define an inner product on X by
$$(f, g) = \int_{-1}^{1} f(s)\overline{g(s)}\, ds\,.$$
   Let $m(s)$ be a continuous function of absolute value 1, that is, $|m(s)| = 1$, $-1 \le s \le 1$. Define M to be multiplication by m:
$$(Mf)(s) = m(s)f(s)\,.$$
   Show that M is unitary.

5. Let A be a linear map of a finite-dimensional complex Euclidean space X.

   (a) A matrix is *normal* if $AA^* = A^*A$. It is unitarily similar to a diagonal matrix if $A = U^*DU$ for a diagonal matrix D and unitary matrix U. Show that these conditions are equivalent.

   (b) Prove that if A is normal then it has a square-root, that is, a matrix B such that $A = B^2$. Is B necessarily normal? Unique?

   (c) Suppose that A is diagonalizable. Prove that A is normal if and only if each eigenvector of A is an eigenvector of A^*.