Read: Lax, Chapter 7, pages 89–100.

1. Let X be an n-dimensional real Euclidean space, and $A\colon X \to X$ a linear map. Define the map $f\colon X \to X$ by $f(x,y) = x^T Ay$. Give (with proof) necessary and suffcient conditions on A for f to be an inner product on X.

2. Express $q(x_1, x_2, x_3) = 3x_1^2 + 8x_1x_2 - 7x_1x_3 + 12x_2^2 - 8x_2x_3 + 6x_3^2$ as $q(x) = x^T Ax$, where A is symmetric.

3. Let
$$M = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix},$$
and let $q(x) = (x, Mx)$. Find an orthogonal matrix P which diagonalizes the quadratic form q.

4. (a) Write the equation $5x_1^2 - 6x_1x_2 + 5x_2^2 = 1$ in the form $x^T Ax = 1$.
   (b) Write $A = P^T DP$, where D is a diagonal matrix and P is orthogonal with determinant 1.
   (c) Sketch the graph of the equation $x^T Dx = 1$ in the x_1x_2-plane.
   (d) Use a geometric argument applied to part (c) to sketch the graph of $x^T Ax = 1$.

5. Repeat the previous problem for the equation $2x_1^2 + 6x_1x_2 + 2x_2^2 = 1$.