Read: Lax, Chapter 10, pages 143–153, and Appendix 4, pages 313–316.

Let U and V be vector spaces over a field K. The *tensor product* of U and V is the set $U \otimes V$ of finite sums of the form

$$\sum c_i(u_i \otimes v_i)\,,$$

where $c_i \in K$, $u_i \in U$, and $v_i \in V$, subject to the following relations:

$$\begin{aligned} c(u \otimes v) &= (cu) \otimes v = u \otimes (cv)\,, \\ (u+u') \otimes v &= u \otimes v + u' \otimes v\,, \\ u \otimes (v+v') &= u \otimes v + u \otimes v'\,. \end{aligned}$$

Elements of the form $u \otimes v$ are called *pure tensors*.

1. Let U, V, and X be vector spaces over a field K. Define a map

$$\tau\colon U \times V \longrightarrow U \otimes V\,, \qquad \tau(u,v) = u \otimes v\,.$$

   (a) Prove that τ is bilinear.

   (b) Prove that for any linear map $A\colon U \otimes V \to X$, the mapping $\alpha := A \circ \tau$ is bilinear from $U \times V$ to X.

   (c) Prove that for any bilinear map $\beta\colon U \times V \to X$, there is a unique linear mapping $A\colon U \otimes V \to X$ such that $\beta = A \circ \tau$. This is the *universal property* of the tensor product, and it says that any bilinear map can be "factored through" it. This is illustrated by the following commutative diagram:

$$\begin{tikzcd} U \times V \arrow[rr, "\beta"] \arrow[dr, "\tau"'] & & X \\ & U \otimes V \arrow[ur, dashed, "A"'] & \end{tikzcd}$$

2. If $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_m\}$ are bases for U and V, respectively, then it is elementary to show that the pure tensors $\{u_i \otimes v_j \mid 1 \le i \le n,\ 1 \le j \le m\}$ span $U \otimes V$. Show that these are linearly independent, and conclude that $\dim(U \otimes V) = (\dim U)(\dim V)$. [*Hint*: Use the canonical basis $\{f_{ij}\}$ of the space of bilinear functions $U \times V \to K$, and use the universal property.]

3. Use the universal property of the tensor product to prove the following results:

   (a) $U \otimes V \cong V \otimes U$ (hint: let $X = V \otimes U$);

   (b) $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$;

   (c) $(U \times V) \otimes W \cong (U \otimes W) \times (V \otimes W)$.