Math 8530: Linear Algebra

4. Matrices

Let $T: X \to U$ be a linear map.

Goal: Encode T as a matrix.

We need to pick:
- a basis $\mathcal{B}_{in} = \{x_1, \dots, x_n\}$ for X ("input basis")
- a basis $\mathcal{B}_{out} = \{u_1, \dots, u_m\}$ for U ("output basis")

Let $\{\ell_1, \dots, \ell_m\}$ be the dual basis in U'.

Next, write

$$
\begin{aligned}
Tx_1 &= a_{11}u_1 + a_{21}u_1 + \dots + a_{m1}u_m \\
Tx_2 &= a_{12}u_1 + a_{22}u_2 + \dots + a_{m2}u_m \\
&\vdots \\
Tx_j &= a_{ij}u_1 + \dots + a_{ij}u_i + \dots + a_{mj}u_m \\
&\vdots \\
Tx_n &= a_{1n}u_1 + a_{2n}u_n + \dots + a_{mn}u_m
\end{aligned}
$$

$\boxed{a_{ij} = \ell_i(Tx_j) = (\ell_i, Tx_j)}$

The matrix of T wrt $\mathcal{B}_{in}$ and $\mathcal{B}_{out}$ is

$$
A = {}_{\mathcal{B}_{in}}[T]_{\mathcal{B}_{out}} := \begin{array}{c} \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{21} & a_{22} & & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ \begin{matrix} Ax_1 & Ax_2 & \quad & A_n \end{matrix} \end{array}
$$

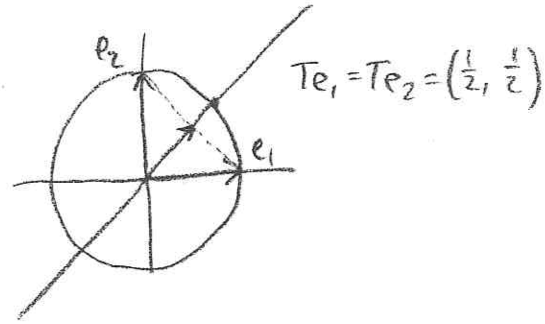
Remarks
- $R_T = \text{Span(col. vectors)}$, also called the "column space."
- $a_{ij} = (\ell_i, Tx_j)$

Ex: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the line $y = x$.

First, consider $\mathcal{B}_{in} = \mathcal{B}_{out} = \{e_1, e_2\}$.

$\boxed{2}$

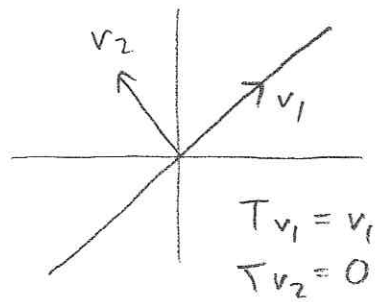
$$Te_1 = \tfrac{1}{2} e_1 + \tfrac{1}{2} e_2$$
$$Te_2 = \tfrac{1}{2} e_1 + \tfrac{1}{2} e_2$$



So $\;{}_{\mathcal{B}_{in}}[T]_{\mathcal{B}_{out}} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

Let's pick a different basis: $\mathcal{B}_{in}' = \mathcal{B}_{out}' = \left\{ v_1 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, v_2 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \right\}$.

$$Tv_1 = \underline{1}\, v_1 + 0 v_2$$
$$Tv_2 = 0 v_1 + 0 v_2 \quad \Rightarrow \quad {}_{\mathcal{B}_{in}'}[T]_{\mathcal{B}_{out}'} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



<u>Remark</u>: If $T: X \to X$ is invertible, then we can <u>always</u> choose $\mathcal{B}_{in}$ and $\mathcal{B}_{out}$ so ${}_{\mathcal{B}_{in}}[T]_{\mathcal{B}_{out}} = I$.

If $\mathcal{B}_{in} = \{v_1, \dots, v_n\}$,

then $\mathcal{B}_{out} := \{Tv_1, \dots, Tv_n\}$

<u>Ex</u>: Let $X = P_2 = \{ c_0 + c_1 x + c_2 x^2 : c_i \in \mathbb{R} \}$ $\qquad \mathcal{B}_{in} = \{1, x, x^2\}$

$U = P_1 = \{ c_0 + c_1 x : c_i \in \mathbb{R} \}$ $\qquad \mathcal{B}_{out} = \{1, x\}$

Let $T = \frac{d}{dx}$, so $T(c_0 + c_1 x + c_2 x^2) = c_1 + 2c_2 x$.

$$T(1) = 0 \cdot 1 + 0x$$
$$T(x) = 1 \cdot 1 + 0x \qquad A = {}_{\mathcal{B}_{in}}[T]_{\mathcal{B}_{out}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$
$$T(x^2) = 0 \cdot 1 + 2x$$

Multiplying matrices (4 ways)

① Rows times columns

$$\text{row } i\ \left[\begin{array}{c} \\ \text{——}\ \ell_i\ \text{——} \\ \\ \end{array}\right]_{m\times n} \left[\begin{array}{c} x_j \\ \Big| \\ \end{array}\right]_{n\times p} = \left[\begin{array}{c} a_{ij} \\ \bullet \\ \\ \end{array}\right]_{m\times p}$$

col j

$$\boxed{a_{ij} = (\ell_i, x_j)}$$

② By columns:

$$\left[\ A\ \right] \overset{B}{\left[\begin{array}{ccc} \Big| & \cdots & \Big| \\ \vec{x}_1 & \cdots & \vec{x}_p \end{array}\right]} = \underbrace{\left[\begin{array}{ccc} \Big| & \cdots & \Big| \\ A\vec{x}_1 & \cdots & A\vec{x}_p \end{array}\right]}_{\text{these columns are linear combinations of cols. of } A.}$$

③ By rows

$$\left[\begin{array}{c} \text{——}\ \ell_1\ \text{——} \\ \vdots \\ \text{——}\ \ell_m\ \text{——} \end{array}\right] \left[\ B\ \right] = \left[\begin{array}{c} \text{——}\ \ell_1 B\ \text{——} \\ \vdots \\ \text{——}\ \ell_m B\ \text{——} \end{array}\right]$$

Each row is a linear combin. of rows of B.

④ Sum of rank 1 matrices: (cols of A) × (rows of B)

$$\underset{n \text{ columns}}{\overset{A}{\left[\begin{array}{ccc} \Big| & \cdots & \Big| \end{array}\right]}} \underset{n \text{ rows}}{\overset{B}{\left[\begin{array}{c} \text{——} \\ \vdots \\ \text{——} \end{array}\right]}} = \sum_{j,k=1}^{n} \underbrace{\vec{a}_j \vec{b}_k^T}_{m\times p \text{ matrix}}$$

<u>Recall:</u> If $T: X \to U$ is linear, and A is the matrix wrt bases $x_1, \dots, x_n$ & $\underbrace{u_1, \dots, u_m}$

with dual basis $l_1, \dots, l_m \in U'$.

[that is, $l_i(u_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$]

Then $\boxed{a_{ij} = (l_i, Tx_j)}$

Thus, to get a_{ij}, where $A = (a_{ij})$:

- Apply T to the j^{th} basis vector of X
- Then apply the i^{th} co-vector in U' to this

Now, consider $T': U' \to X'$. Find matrix (a'_{ij})

- Apply T' to the j^{th} basis vector of U'
- Then apply the i^{th} co-vector in $X'' = X$ to this:

$$a'_{ij} = (\hat{x}_i, T'l_j) \Rightarrow (T'l_j, x_i) = (l_j, Tx_i) = a_{ji}$$

in X'' identify $X'' = X$

* Thus, the matrix (a'_{ij}) for T' satisfies $a'_{ij} = a_{ji}$.

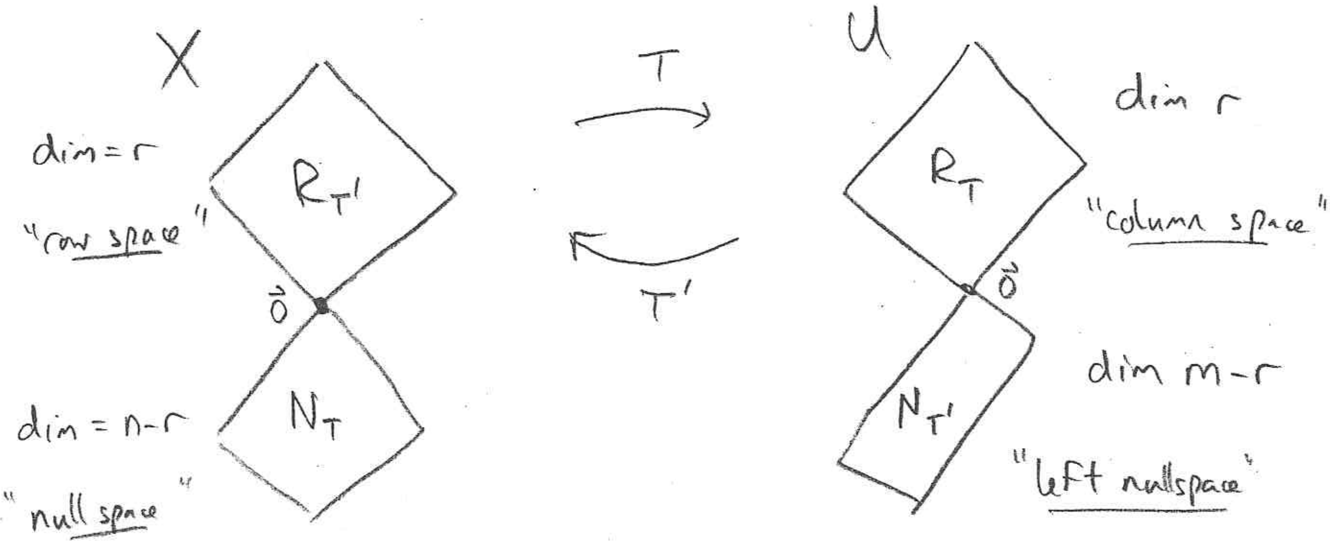
Denote this matrix by A^T. Obtained by swapping rows with columns.

Recall: The range R_T consists of all linear combinations of the column vectors. The dimension of this space is called the column rank of T. The row rank is defined similarly.

Remarks: (i) row rank of T is $\dim R_{T'}$

(ii) By thm 3.6 ($\dim R_T = \dim R_{T'}$), row rank and column rank are equal.

"Cartoon" of this: $T : X \to U$



Prop: T is a bijection when restricted $R_{T'} \to R_T$.

Proof: Only need to show it's 1-1.

Suppose $Tl_1 = Tl_2 \Rightarrow T(l_1 - l_2) = 0 \Rightarrow l_1 - l_2 \in N_T$

But $l_1 - l_2$ also in $R_{T'}$ (since its a subspace).

$\Rightarrow l_1 - l_2 = 0$, since $X = R_{T'} \oplus N_T$. $\square$

Change of basis

Let $T: X \to U$ be linear, $x_1, \dots, x_n$ & $u_1, \dots, u_m$ be bases.

Since $\dim X = n$, $\dim U = m$, we have $X \simeq \mathbb{R}^n$, $U \simeq \mathbb{R}^m$.

That is, we have isomorphisms:

$$B: X \longrightarrow \mathbb{R}^n \qquad C: U \longrightarrow \mathbb{R}^m$$
$$x_i \longmapsto e_i \qquad u_i \longmapsto e_i$$

Putting this together, we can choose isomorphisms such as these to get a linear map $CTB^{-1}: \mathbb{R}^n \to \mathbb{R}^m$

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "B"'] & U \arrow[d, "C"] \\ \mathbb{R}^n \arrow[r, "CTB^{-1}"] & \mathbb{R}^m \end{tikzcd}$$

If $T: X \to X$ then we can take $x_i = u_i$ & $B = C$ and get a matrix $M = BTB^{-1}$.

Suppose we change the isomorphism B. In particular, let $A: X \to \mathbb{R}^n$ be another isomorphism, and let N be the matrix wrt this basis, i.e., $N = ATA^{-1}$

We have $N = ATA^{-1} = AB^{-1}BTB^{-1}BA^{-1} = SMS^{-1}$

where $S = AB^{-1}$, which is invertible.

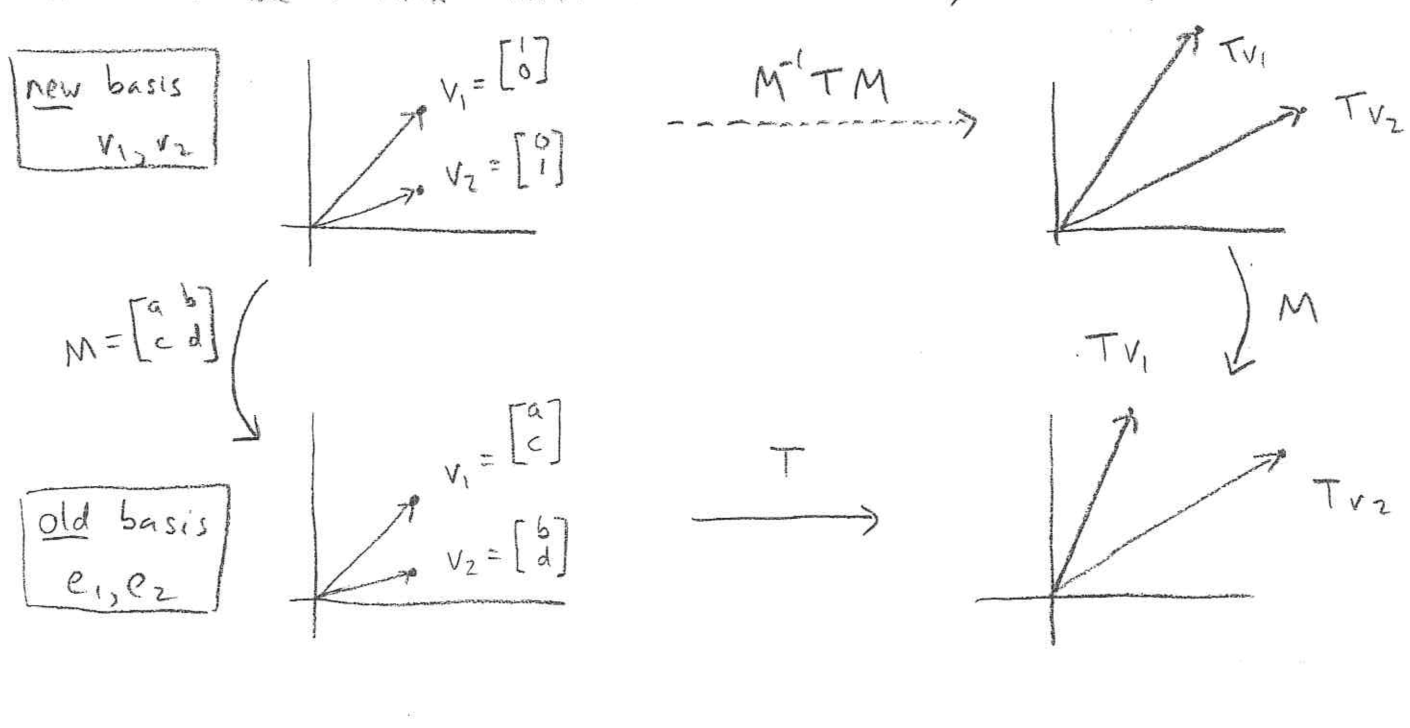
$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "B"'] & X \arrow[d, "B"] \\
\mathbb{R}^n \arrow[r, "M=BTB^{-1}"] \arrow[d, "B^{-1}"'] \arrow[dd, bend right=60, "S"'] & \mathbb{R}^n \arrow[d] \arrow[dd, bend left=60, "S"] \\
X \arrow[r] \arrow[d, "A"'] & X \arrow[d] \\
\mathbb{R}^n \arrow[r, "N=SMS^{-1}"] & \mathbb{R}^n
\end{tikzcd}$$

Two square matrices M, N related by conjugation (e.g., $N = SMS^{-1}$) are said to be <u>conjugate</u>.

$\star$ Similar matrices describe the <u>same</u> mapping of a space into itself, but using different bases. Thus (as we'll see later) similar matrices share the same intrinsic properties.

<u>Picture of this:</u> Consider a 2×2 matrix wrt the standard unit basis vectors $e_1, e_2 \in \mathbb{R}^2$.

What is the matrix w.r.t. a different basis, $v_1 = \begin{bmatrix} a \\ c \end{bmatrix}$, $v_2 = \begin{bmatrix} b \\ d \end{bmatrix}$?



<u>Remark:</u> We can write any $n\times n$ matrix A in <u>block form</u>:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{where} \quad A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)\begin{array}{l} \}\, k \text{ rows} \\ \}\, n-k \text{ rows} \end{array}$$

$$\qquad\qquad\qquad\qquad\underbrace{\qquad\quad}_{k \text{ columns}}\ \underbrace{\qquad\quad}_{n-k \text{ columns}}$$

Addition and multiplication of block matrices "works out" just as if the blocks were entries.

<u>Def:</u> The square matrix $I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ is the <u>identity</u> matrix.

<u>Def:</u> A square matrix $T = (t_{ij})$ for which $t_{ij} = 0$ for $i > j$ (resp. $i < j$) is called <u>upper triangular</u> (resp. lower triangular).

<u>Systems of equations</u>

Matrices can be used to effectively express and solve systems of equations.

A system $\sum_{i=1}^{n} t_{ij} x_i = u$ for $j = 1, \dots, n$ may have a unique sol'n, many solutions, or no solutions.

<u>Example:</u> $x_1, \dots, x_4$ unknowns:

$$\begin{aligned} x_1 + x_2 + 2x_3 + 3x_4 &= u_1 \\ x_1 + 2x_2 + 3x_3 + x_4 &= u_2 \\ 2x_1 + x_2 + 2x_3 + 3x_4 &= u_3 \\ 3x_1 + 4x_2 + 6x_3 + 2x_4 &= u_4 \end{aligned} \quad \rightsquigarrow \quad \underbrace{\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 3 \\ 3 & 4 & 6 & 2 \end{pmatrix}}_{M} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}}_{\vec{u}}$$

System can be solved by <u>Gaussian elimination</u>:

Subtract multiples of Eq 1 from last 3 eq'ns to eliminate x_1:

$$x_2 + x_3 - 2x_4 = u_2 - u_1$$
$$-x_2 - 2x_3 - 3x_4 = u_3 - 2u_1$$
$$x_2 \quad -7x_4 = u_4 - 3u_1$$

Use 1st eq'n to eliminate x_2 from last 2:

$$-x_3 - 5x_4 = u_3 + u_2 - 3u_1$$
$$-x_3 - 5x_4 = u_4 - 2u_2 - 2u_1$$

Subtract these equations to eliminate x_3 (& by chance, x_4 too)

$$\boxed{0 = u_4 - u_3 - 2u_2 + u_1}$$

This is a <u>necessary</u> & <u>sufficient</u> condition for our original system to have a sol'n. It can be written as $(l, u) = 0$, where $l = (1, -2, -1, 1)$.

That is, $(1, -2, -1, 1)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = u_1 - 2u_2 - u_3 + u_4 = 0.$

Since $Mx = u$ and $lu = 0$, we must have

$lMx = lu = 0$ for all $x \in \mathbb{R}^4$, thus $lM = 0$.

* In general, if $Mx = u$ is a system of equations then a sol'n is described by a linear function $l \in X'$, or equivalently, a row vector for which $lM = 0$.