7. Partial differential equations

Let $u(x,t)$ be a 2-variable function. A <u>partial differentiable equation</u> (PDE) is an equation involving u, x, t, and the partial derivatives of u.

<u>Example</u>: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2}$ (or just $u_t = u_{xx}$)

ODE's have a unifying theory of existence & uniqueness of solutions.

PDE's have no such theory.

PDE's arise from physical phenomena & modeling.

<u>Heat equation</u>: $\rho(x)\sigma(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right)$, where

$u(x,t)$ = temperature of a bar at position x & time t.

$\rho(x)$ = density of the bar at position x

$\sigma(x)$ = specific heat at position x

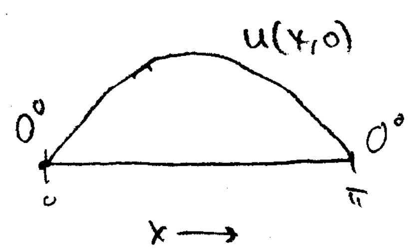
$\kappa(x)$ = thermal conductivity at position x.

* We'll assume that the bar is uniform (i.e, ρ, σ, κ <u>constant</u>).

In this case, the heat equation becomes:

$$\boxed{\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}} \quad \text{where } c^2 = \frac{\kappa}{\rho\sigma}.$$

<u>Example 1(a)</u>: Let $u(x,t)$ = temp. of a bar of length π, insulated along the sides, whose ends are kept at $0°$ for all time (<u>Boundary</u> conditions) and $u(x,0) = x(\pi - x)$ (<u>Initial</u> condition)



Thus, we have the following <u>initial/boundary value problem</u>.

$$u_t = c^2 u_{xx}, \quad u(0,t)=0, \quad u(\pi,t)=0, \quad u(x,0) = x(\pi - x)$$

<u>Note</u>: This is <u>homogeneous</u> and <u>linear</u>, i.e., if u_1 & u_2 are solutions, then so is $c_1 u_1 + c_2 u_2$ (<u>superposition</u>)

Let's <u>solve</u> this!

* Assume $\boxed{u(x,t) = f(x)\, g(t)}$ "Separation of variables"

$$u_t = f(x)\, g'(t) \quad \text{and} \quad u_{xx} = f''(x)\, g(t).$$

Boundary conditions: $u(0,t) = f(0)\, g(t) \Rightarrow f(0) = 0$

$u(\pi,t) = f(\pi)\, g(t) \Rightarrow f(\pi) = 0.$

Plug back in & solve for f & g:

$$u_t = c^2 u_{xx} \Rightarrow f(x)\, g'(t) = c^2 f''(x)\, g(t)$$

$$\Rightarrow \frac{g'(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)} = \lambda \quad \text{The "eigenvalue equation".}$$

Doesn't depend on x

Doesn't depend on t

Therefore, this must be <u>constant</u>.

Now, we have 2 ODEs: $\dfrac{g'(t)}{c^2 g(t)} = \lambda$, $\dfrac{f''(x)}{f(x)} = \lambda$

i.e, $g' = c^2 \lambda g$ and $f'' = \lambda f$, $f(0) = 0$, $f(\pi) = 0$

<u>Solve for g</u>: $\boxed{g(t) = A e^{c^2 \lambda t}}$ ✓

Solve for f: $f'' = \lambda f$, $f(0) = f(\pi) = 0$

Case 1: $\boxed{\lambda = 0}$: $f'' = 0 \Rightarrow f(x) = ax + b$.

$f(x) = 0 \Rightarrow a = 0$. $f(\pi) = 0 \Rightarrow b = 0 \Rightarrow f(x) = 0$.

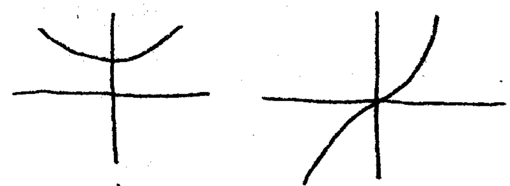
Case 2: $\boxed{\lambda > 0}$. $f'' = w^2 f$ $(\lambda = w^2)$

$f(x) = C_1 e^{wx} + C_2 e^{-wx}$ or $f(x) = A\cosh wx + B\sinh wx$.

$f(0) = A = 0 \Rightarrow f(x) = B\sinh wx$

$f(\pi) = B\sinh w\pi = 0 \Rightarrow B = 0$

(Recall: $\cosh 0 = 1$, $\sinh = 0$)



cosh x sinh x

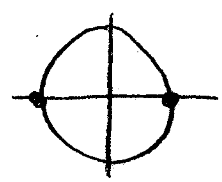
Case 3: $\boxed{\lambda < 0}$ $f'' = -w^2 f$ $(\lambda = -w^2)$

$f(x) = a\cos wx + b\sin wx$.

$f(0) = a = 0 \Rightarrow f(x) = b\sin wx$.

$f(\pi) = b\sin w\pi = 0 \Rightarrow w\pi = n\pi$

$\Rightarrow \boxed{w = n}$



$\sin w\pi = 0$ iff $w\pi = n\pi$

Therefore, $\boxed{f(x) = b\sin nx}$, for any integer n.

* In summary, for any fixed choice $\lambda = -n^2$, we have a solution $u_n(x,t) = f_n(x)\, g_n(t)$, where $g_n(t) = A_n e^{-c^2 n^2 t}$

$f_n(t) = B_n \sin nx$.

Thus, $\boxed{u_n(x,t) = b_n e^{-c^2 n^2 t} \sin nx}$ is a solution for any n.

[Here, we just "absorb" the constants into one constant, b_n].

By superposition, any linear combination of sol'ns is also a sol'n.

Thus, the general solution is $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$, i.e.,

$$(*) \quad \boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \; e^{-c^2 n^2 t}}$$

Now, let's solve the initial value problem: $u(x,0) = x(\pi - x)$.

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx = x(\pi - x) \quad \text{on } [0, \pi].$$

To solve for the b_n's, we must write $x(\pi - x)$ as a

Fourier sine series.

Recall: $b_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin nx \, dx = \frac{4}{\pi n^3}\left(1 - (-1)^n\right)$ (See Hw)

Thus, $u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1-(-1)^n\right) \sin nx$

$$\Rightarrow b_n = \frac{4}{\pi n^3}\left(1 - (-1)^n\right).$$

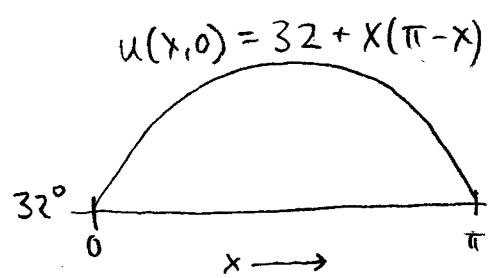
Our particular solution to the initial value problem is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1-(-1)^n\right) \sin nx \; e^{-c^2 n^2 t}}$$

Remark: The steady-state solution is $\lim_{t \to \infty} u(x,t) = 0$.

(because $e^{-c^2 n^2 t} \to 0$ as $t \to \infty$).

Example 1(b): Consider the same physical situation, but now, say that the boundaries are held fixed at 32° (initial cond. adjusted accordingly).



We now have the following initial/boundary value problem:

$$u_t = c^2 u_{xx}, \quad u(0,t) = u(\pi,t) = 32, \quad u(x,0) = x(\pi - x) + 32$$

Question: What's the solution? (i.e., how does it differ from the previous example?)

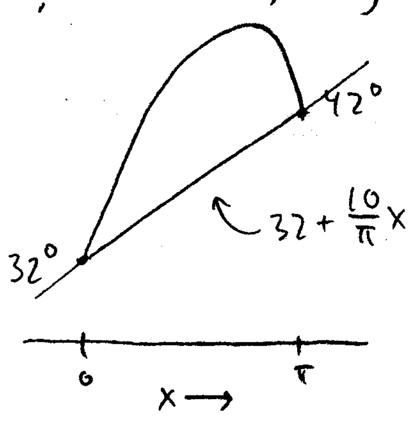
Answer: $\boxed{u(x,t) = \underline{\underline{32}} + \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1-(-1)^n\right) \sin nx \; e^{-c^2 n^2 t}}$

Motivation: This is exactly the same as the previous problem, had we proclaimed 32° to be 0° (say, in a "new" temperature system).

Remark: $\lim_{t\to\infty} u(x,t) = 32$ is the steady-state solution.

Moral: $u(x,t) = u_h(x,t) + u_p(x,t)$, where $u_h(x,t)$ is the solution to the homogeneous eq'n (including boundary conditions), and $u_p(x,t)$ is any particular solution (e.g. steady-state sol'n).

Example 1(c): Consider the same physical situation, but now, say that the left-hand boundary is fixed at 32°, and the right-hand boundary is fixed at 42°. (& init. conds. adjusted accordingly.



We now have the following initial/boundary value problem:

$$u_t = c^2 u_{xx}, \quad u(0,t) = \underline{\underline{32}}, \quad u(\pi,t) = \underline{\underline{42}}$$

$$u(x,0) = 32 + \frac{10}{\pi}x + x(\pi - x)$$

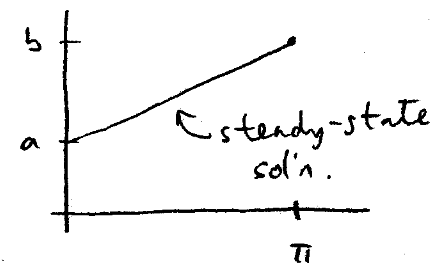
The solution (not surprisingly) is

$$\boxed{u(x,t) = \underbrace{32 + \frac{10}{\pi} x}_{\text{steady-state solution}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin nx \, e^{-c^2 n^2 t}}_{\text{"homogeneous" part, i.e., sol'n if the boundary conditions were both 0.}}}$$

<u>Summary</u>: To solve $u_t = c^2 u_{xx}$, $u(0,t) = a$, $u(\pi,t) = b$, $u(x,0) = h(x)$,

First solve the related problem where $a = b = 0$, then add this to the steady-state solution, which will clearly (<u>why</u>?) be $u_{ss}(x,t) = a + \frac{b-a}{\pi} x$.

i.e, $\boxed{u(x,t) = u_{0,0}(x,t) + u_{ss}(x,t)}$

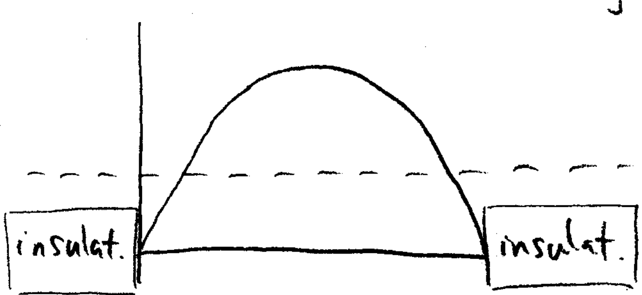


* Convince yourself why this makes sense physically..

$$\lim_{t \to \infty} u_{0,0}(x,t) = 0, \quad \text{so} \quad \lim_{t \to \infty} u(x,t) = u_{ss}(x,t) = a + \frac{b-a}{\pi} x.$$

<u>Example 2</u>: Same situation as Example 1, but with <u>different</u> boundary conditions.

$$u_t = c^2 u_{xx}, \qquad \underbrace{u_x(0,t) = u_x(\pi,t) = 0}_{\text{Represents insulated endpoints, through which no heat can pass}}, \qquad u(x,0) = x(\pi - x).$$



steady-state sol'n = average temp.
(as we'll see, this is $\frac{a_0}{2}$!)

Remark: The only difference between this, and Example 1a, is

$$u_x(0,t) = u_x(\pi,t) = 0 \Rightarrow f'(0) = f'(\pi) = 0 \quad (\text{vs. } f(0) = f(\pi) = 0).$$

- $g(t)$ is the same as before: $g(t) = A_n e^{-c^2n^2t}$.
- $f(x)$ has different boundary conditions: $f'' = \lambda f$, $f'(0) = f'(\pi) = 0$.

This has sol'n $f(x) = a\cos wx + b\sin wx$

$f'(0) = bw = 0 \Rightarrow b = 0$

$f'(\pi) = aw\sin w\pi = 0 \Rightarrow w\pi = n\pi \Rightarrow w = n$ (as before).

$\Rightarrow \boxed{f_n(x) = a_n\cos nx}$ and $\boxed{g_n(t) = A_n e^{-c^2n^2t}}$ for $n \geq 0$.

Thus, the general solution becomes:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} f_n(x)\,g_n(t)$$

(Note: when $n=0$, f_n & g_n are constants, not necessarily zero).

$$\Rightarrow \boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx\, e^{-c^2n^2t}}$$

Now, let's solve the initial value problem: $u(x,0) = x(\pi - x)$.

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx = \underbrace{x(\pi - x)}_{\text{We must express this as a Fourier cosine series.}} \quad \text{on } [0,\pi].$$

Recall: (HW) $a_0 = \frac{\pi^2}{3}$, $a_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos nx\,dx = \frac{2}{n^2}(1-(-1)^n)$.

Thus, $u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}(1-(-1)^n)\cos nx$

$\Rightarrow \frac{a_0}{2} = \frac{\pi^2}{6}$ and $a_n = \frac{2}{n^2}(1-(-1)^n)$, so the solution to the IVP is

$$\boxed{u(x,t) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}(1-(-1)^n)\cos nx\, e^{-c^2n^2t}}$$

Wave equation: $u_{tt} = c^2 u_{xx}$

Motivation: Consider the following PDE: $\boxed{\dfrac{\partial u}{\partial t} - c\dfrac{\partial u}{\partial x} = 0}$ $(*)$

Let $f(x)$ be any one-variable function, and set

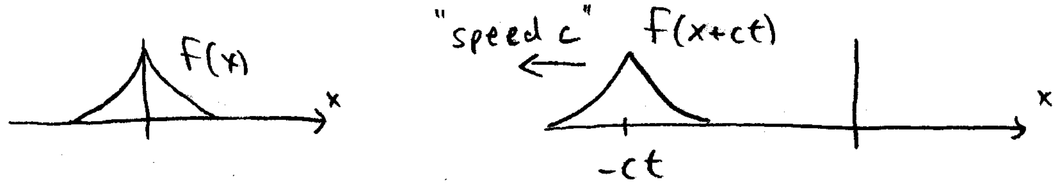
$u(x,t) = f(x+ct)$. Chain rule $\Rightarrow$ $u_x(x,t) = f'(x+ct)$

$u_t(x,t) = c\,f'(x+ct)$

Note: $u_t - c u_x = c\,f'(x+ct) - c\,f'(x+ct) = 0$ ✓

i.e., $f(x+ct)$ is a solution to the PDE in $(*)$.

Picture of this:



As t increases, $u(x,t) = f(x+ct)$ is a traveling wave, to the left, at speed c.

Next, consider the PDE $\boxed{\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0}$ $(**)$

Let $g(x)$ be any one-variable function, and set

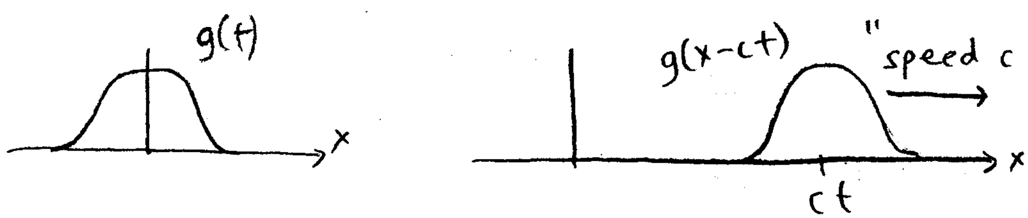
$u(x,t) = g(x-ct)$ Chain rule $\Rightarrow$ $u_x(x,t) = g'(x-ct)$

$u_t(x,t) = -c\,g'(x-ct)$

Note: $u_t + c u_x = -c\,g'(x-ct) + c\,g'(x-ct) = 0$ ✓

i.e., $g(x-ct)$ is a solution to the PDE in $(**)$.

Picture of this:



Now, let $f(x)$ & $g(x)$ be *any* two one-variable functions. Consider the PDE:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = \boxed{\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0} \quad (***)$$

Check: $u(x,t) = f(x+ct) + g(x-ct)$ is a solution.

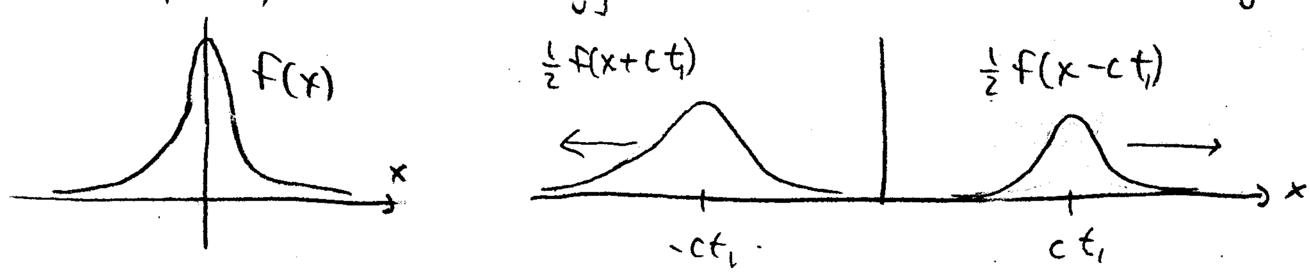
Consider the following initial value problem:

$u_{tt} = c^2 u_{xx}$ (the PDE in (***))

$u(x,0) = f(x)$ Initial displacement, or initial wave.

$u_t(x,0) = 0$ Initial velocity (vertical, pointwise).

Picture of this: Start with a stationary wave in the ocean, on a string, etc. Then "let go" at time $t=0$. It should disperse, "half the energy to the left, half to the right."



Initially; $t=0$. At time $t=t_1$ in the future.

The solution to this IVP is

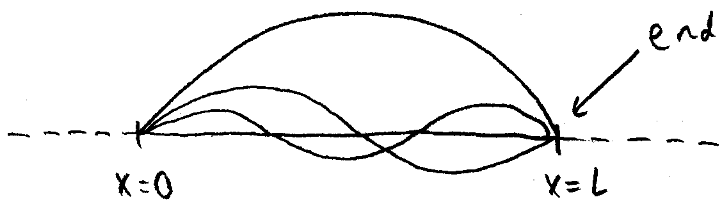
$$u(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct).$$

This matches our physical intuition!

Big idea: $\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \text{ is the wave equation}}$

Now, suppose we want to model vibrations (waves) on a finite string/wire of length L.

We need to impose <u>boundary conditions</u>.



Let $u(x,t)$ be the (vertical) displacement at position x & time t.

Fixed endpoints $\Rightarrow$ $u(0,t)=0$ and $u(L,t)=0$.

We must specify the initial wave: $u(x,0)=h_1(x)$

and initial (vertical) velocity @ x: $u_t(x,0)=h_2(x)$.

Together, we get an initial/boundary value problem for the <u>wave equation</u>:

$$\boxed{\begin{array}{l} u_{tt}=c^2u_{xx}, \quad u(0,t)=0, \quad u(L,t)=0 \\ u(x,0)=h_1(x), \quad u_t(x,0)=h_2(x) \end{array}}$$

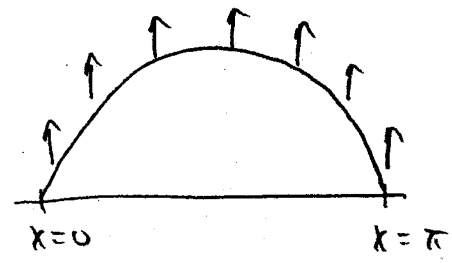
<u>Remark</u>: This is essentially the 1-dimensional analog of the (0-dimensional) equation of harmonic motion: $y''=-\omega^2 y$, i.e, each point on the wave acts like a mass-spring system.

We can solve this PDE using <u>separation of variables</u>, just like we did for the heat equation.

There are only a few slight differences.

Example 3: Consider the PDE

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 0, \quad u(\pi,t) = 0 \\ u(x,0) = x(\pi - x), \quad u_t(x,0) = 1 \end{cases}$$



picture at $t=0$.

Separation of variables: Assume $u(x,t) = f(x)\,g(t)$ & plug back in:

$$u_{tt} = f g'', \quad u_{xx} = f'' g \Rightarrow f g'' = c^2 f'' g \Rightarrow \boxed{\frac{f''}{f} = \frac{g''}{c^2 g} = \lambda}$$

"Zero-boundary conditions:" $u(0,t) = f(0)\,g(t) = 0 \Rightarrow f(0) = 0$

$u(\pi,t) = f(\pi)\,g(t) = 0 \Rightarrow f(\pi) = 0$.

The "eigenvalue equation" gives us 2 ODEs:

$$\begin{cases} f'' = \lambda f, \quad f(0) = f(\pi) = 0 \\ g'' = c^2 \lambda g \end{cases}$$

$f(x)$: Same as in heat equation: $\lambda = -n^2$, $\boxed{f_n(x) = b_n \sin nx}$

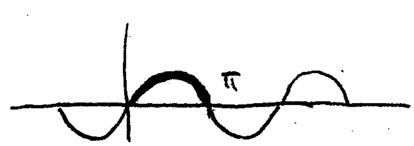
$g(t)$: $g'' = -c^2 n^2 g \Rightarrow \boxed{g_n(t) = a_n \cos(cnt) + b_n \sin(cnt)}$

Thus, the general solution, by superposition, is

$$u(x,t) = \sum_{n=0}^{\infty} \underbrace{f_n(x)\,g_n(t)}_{u_n(x,t)} = \boxed{\sum_{n=1}^{\infty} \left(a_n \cos(cnt) + b_n \sin(cnt)\right) \sin nx}$$

Finally, use (both) initial conditions.

(i) $u(x,0) = x(\pi - x)$



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$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx = x(\pi - x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1-(-1)^n\right) \sin nx$$

(The Fourier sine series of $x(\pi-x)$)

$$\Rightarrow \boxed{a_n = \frac{4}{\pi n^3}\left(1-(-1)^n\right)}$$

(ii) $u_t(x,0) = 1$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(-cna_n \sin(cnt) + cnb_n \cos(cnt)\right) \sin(nx)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} cnb_n \sin(nx) = 1 = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1-(-1)^n\right) \sin(nx)$$

(The Fourier sine series of 1).

$$\Rightarrow cnb_n = \frac{2}{n\pi}\left(1-(-1)^n\right) \Rightarrow \boxed{b_n = \frac{2}{cn^2\pi}\left(1-(-1)^n\right)}$$

The solution to this initial/boundary value problem for the wave equation is thus

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos(cnt) + b_n \sin(cnt)\right) \sin nx, \quad \text{i.e.,}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[\frac{4}{\pi n^3}\left(1-(-1)^n\right) \cos(cnt) + \frac{2}{\pi c n^2}\left(1-(-1)^n\right) \sin(cnt)\right] \sin nx}$$

PDE's in higher dimensions

In 2 (spatial) dimensions, the heat & wave equations are

* Heat equation: $u_t = c^2(u_{xx} + u_{yy})$
* Wave equation: $u_{tt} = c^2(u_{xx} + u_{yy})$

More generally, let $u(x_1, \dots, x_n, t)$ be a function of n spatial variables.

Def: The Laplacian of u is $\nabla \cdot \nabla u = \boxed{\nabla^2 u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}}$

Recall that $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$, so $\nabla \cdot \nabla = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$.

In n dimensions, our familiar PDE's are:

* Heat equation: $u_t = c^2 \nabla^2 u$
* Wave equation: $u_{tt} = c^2 \nabla^2 u$

(Note: Sometimes the Laplace operator ∇^2 is written Δ).

Steady-state solutions occur for the heat equation, but not for the wave equation (heat diffuses, waves propagate).

Remark: "Steady-state" means that $u_t = 0$. Solutions to the heat equation approach this steady-state solution because "eventually, the temperature doesn't change wrt time."

* Thus, all steady-state solutions satisfy $0 = u_t = c^2 \nabla^2 u$,

i.e., $\boxed{\nabla^2 u = 0} \Rightarrow \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = 0.$

Def: A function u is <u>harmonic</u> if $\nabla^2 u = 0$.

Example: $f(x, y) = x^2 - y^2$ is harmonic:

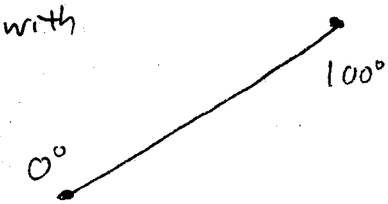
$$f_{xx} = 2, \quad f_{yy} = -2 \quad \Rightarrow \quad \nabla^2 f = f_{xx} + f_{yy} = 2 - 2 = 0 \checkmark$$

Visualizing harmonic functions:

Big idea: Harmonic functions are "as flat as possible."

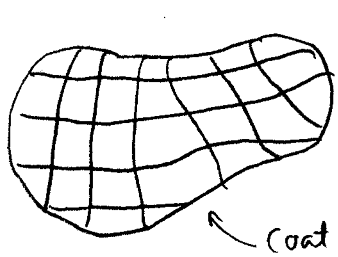
Harmonic functions; $\nabla^2 u = 0$	$\xleftrightarrow{\text{same}}$ (set $u_t = 0$)	Steady-state sol'ns to the heat equation	$\xleftrightarrow{\text{same}}$ (physical intuition)	"Flat" functions; "plastic wrap surfaces" (see below)

In 1D: Consider the temperature $u(x,t)$ of a bar, with $u(0,t) = 0$, $u(L,t) = 100$. The steady-state solution satisfies $0 = u_t = c^2 u_{xx}$, so it is a straight line, regardless of initial condition.



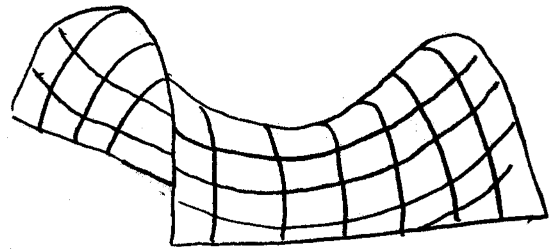
Physical interpretation: Stretch out plastic wrap over a bent circular wire, as tightly as possible.

* The surface is a <u>harmonic function</u>!



Fact: If f is harmonic, then for <u>any</u> closed bounded region R, f achieves its min & max values on the <u>boundary</u>, ∂R.

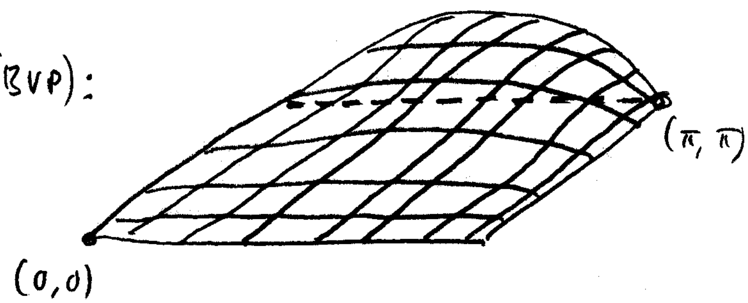
<u>Example:</u> $f(x) = x^2 - y^2$.



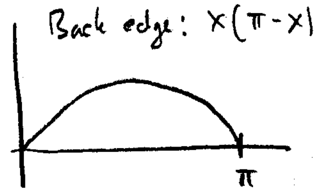
Picture cutting this surface (a saddle) with a "cookie cutter." The max & min points will be on the boundary, i.e, there are no interior local max or mins (it's "flat"!).

The PDE $\nabla^2 u = 0$ is called <u>Laplace's equation</u>.

<u>Example 1(a):</u> Let $u(x,y)$ be a 2-variable function defined for $0 \le x, y \le \pi$ that satisfies the following boundary value problem (BVP):



$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = u(\pi, y) = u(x,0) = 0 \\ u(x,\pi) = x(\pi - x) \end{cases}$$



* <u>Physical situation:</u> $u(x,y)$ is the steady-state solution of the 2D heat equation, where 3 sides are fixed at $0°$, and one at $u(x,\pi) = x(\pi - x)$.

<u>Let's solve this!</u> (Again, by separation of variables).

* Assume $u(x,y) = X(x)\,Y(y)$:

$$u_{xx} = X''Y, \qquad u_{yy} = XY''$$

Use "0-boundary" conditions:
$$u(0,y) = X(0)Y(y) = 0 \Rightarrow X(0) = 0$$
$$u(\pi,y) = X(\pi)Y(y) = 0 \Rightarrow X(\pi) = 0$$
$$u(x,0) = X(x)Y(0) = 0 \Rightarrow Y(0) = 0.$$

<u>Note:</u> The 4^{th} boundary condition $u(x,\pi)$ isn't useful now.

Plug back into the PDE: independent of y independent of x.

$$u_{xx} + u_{yy} = X''Y + XY'' = 0 \Rightarrow \boxed{\frac{X''}{X} = \frac{-Y''}{Y} = \lambda}$$ "Eigenvalue eq'n"

We now have 2 ODE^s: (i) $X'' = \lambda X, \quad X(0) = X(\pi) = 0$

(ii) $Y'' = -\lambda Y, \quad Y(0) = 0$

Let's solve these:

(i) We've done this before: $\boxed{\lambda = -n^2}$, $\boxed{X_n(x) = b_n \sin nx}$

(ii) $Y'' = n^2 Y, \quad Y(0) = 0.$

$Y_n(y) = A_n \cosh ny + B_n \sinh ny$ (This will be easier than $C_1 e^{ny} + C_2 e^{-ny}$).

$Y_n(0) = A_n = 0 \Rightarrow \boxed{Y_n = B_n \sinh ny}$

The general solution is $u(x, y) = \sum_{n=0}^{\infty} X_n(x) Y_n(y)$,

i.e, $\boxed{u(x,y) = \sum_{n=1}^{\infty} b_n \sin nx \sinh ny}$

Finally, use the 4^{th} boundary condition (plug in $y = \pi$).

$$u(x,\pi) = \sum_{n=1}^{\infty} (b_n \sinh n\pi) \sin nx = x(\pi - x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}(1-(-1)^n)\sin nx.$$

Equate coefficients: $b_n \sinh n\pi = \frac{4}{\pi n^3}(1-(-1)^n) \Rightarrow b_n = \frac{4(1-(-1)^n)}{\pi n^3 \sinh n\pi}$

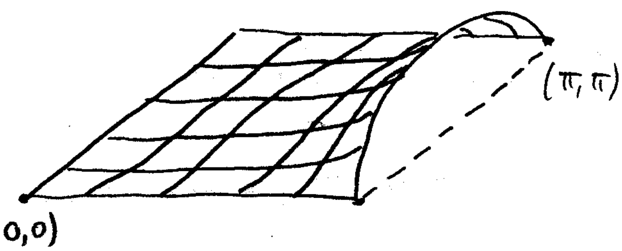
Therefore, the particular solution to the boundary value problem

$$u_{xx} + u_{yy} = 0, \quad u(0,y) = u(\pi, y) = u(x,0) = 0, \quad u(x,\pi) = x(\pi - x).$$

is $\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3 \sinh n\pi} \sin nx \sinh ny.}$

Example 1(b): Consider the following boundary value problem (BVP)

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = u(x,\pi) = u(0,y) = 0. \\ u(\pi, y) = y(\pi - y). \end{cases}$$

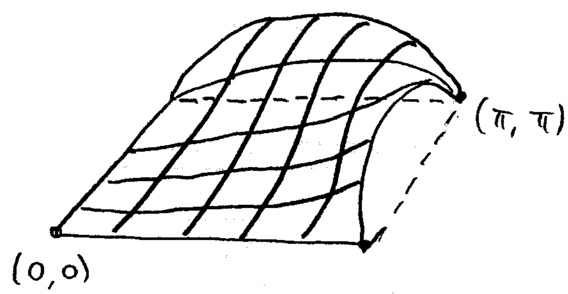


* This is *exactly* the same problem as Example 1(a), but the roles of x & y are reversed.

Thus, by symmetry, the solution is

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3 \sinh n\pi} \sinh nx \ \sin ny}$$

Example 1(c): The following BVP is a "superposition" of Examples 1(a) & 1(b):



$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = u(0,y) = 0 \\ u(x,\pi) = x(\pi - x), \quad u(\pi, y) = y(\pi - y) \end{cases}$$

Not surprisingly, the solution to this BVP is the *sum* of the solutions to Examples 1(a) & 1(b):

$$u(x,y) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3 \sinh n\pi} \Big[(\sin nx + \sinh ny) + (\sinh nx + \sin ny) \Big]$$

Think: Why does this make sense physically, in terms of steady-state heat distributions?

Heat equation in 2D: $u_t = c^2(u_{xx} + u_{yy})$

To solve it (with initial & boundary conditions):

(i) Find the steady-state sol'n first, i.e., solve Laplace's eq'n; $\nabla^2 u = 0$ subject to the same boundary conditions

(ii) Add this to the solution of the homogeneous equation where the boundary conditions are set to zero, but with the same initial condition. We'll see how to do this next.

Example 2(a): let $u(x,y,t)$ = temp. of a square region, $0 \le x, y \le \pi$, subject to

$u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0$ (Boundary fixed at $0°$).

$u(x,y,0) = 2\sin x \sin 2y + 3\sin 4x \sin 5y$ (Initial heat distribution).

Solve by separation of variables: Assume the solution has the form $\boxed{u(x,y,t) = f(x,y)\, g(t).}$

function of position — $f(x,y)$; function of time — $g(t)$.

Note: $u_{xx} = f_{xx}\, g, \quad u_{yy} = f_{yy}\, g, \quad u_t = f g'$

Use "zero-boundary conditions":

$$u(0,y,t) = f(0,y)\, g(t) = 0 \implies f(0,y) = 0$$
$$u(\pi,y,t) = f(\pi,y)\, g(t) = 0 \implies f(\pi,y) = 0$$
$$u(x,0,t) = f(x,0)\, g(t) = 0 \implies f(x,0) = 0$$
$$u(x,\pi,t) = f(x,\pi)\, g(t) = 0 \implies f(x,\pi) = 0$$

Plug $u = fg$ back into the PDE:

$$u_t = c^2(u_{xx} + u_{yy}) \Rightarrow f g' = c^2 f_{xx} g + c^2 f_{yy} g$$

$$\Rightarrow \frac{g'}{c^2 g} = \frac{f_{xx} + f_{yy}}{f} = \frac{\nabla^2 f}{f} = \lambda$$ "Eigenvalue equation."

We get 2 equations:

$$\boxed{\begin{array}{ll} \text{(i)} & \nabla^2 f = \lambda f, \quad f(0,y) = f(\pi,y) = f(x,0) = f(x,\pi) = 0 \\ \text{(ii)} & g' = c^2 \lambda g \end{array}}$$ ← PDE ← ODE

(ii) Solve for g: (it's easier): $g(t) = A e^{c^2 \lambda t}$

(i) Solve for f: $f_{xx} + f_{yy} = \lambda F$ "Helmholz equation."

Separate variables! Assume $f(x,y) = X(x) Y(y) \Rightarrow f_{xx} = X''Y$, $f_{yy} = XY''$

Plug back in: $X''Y + XY'' = \lambda XY \Rightarrow \frac{X''Y}{XY} + \frac{XY''}{XY} = \lambda$

$$\Rightarrow \boxed{\frac{X''}{X} + \frac{Y''}{Y} = \lambda}$$

Rewrite as $\underbrace{\frac{X''}{X}}_{\text{depends only on } x} = \underbrace{\lambda - \frac{Y''}{Y}}_{\text{depends only on } y} = \mu$ ← must be constant!

We get 2 ODEs: $X'' = \mu X$, $\quad Y'' = (\overbrace{\lambda - \mu}^{\text{call this } \nu})Y \Rightarrow \lambda - \mu = \nu$

$Y'' = \nu Y$ $\qquad \boxed{\lambda = \nu + \mu}$

ODE 1: $X'' = \mu X$, $X(0) = X(\pi) = 0 \Rightarrow \boxed{X_n(x) = b_n \sin nx, \quad \mu = -n^2}$

ODE 2: $Y'' = \nu Y$, $Y(0) = Y(\pi) = 0 \Rightarrow \boxed{Y_m(y) = B_m \sin my, \quad \mu = -m^2}$

Recall: $\boxed{\lambda = \nu + \mu = -(n^2+m^2)}$. Thus, for each pair m & n, we have

solutions $\boxed{f_{nm}(x,y) = b_{nm} \sin nx \sin my}$, $\boxed{g_{nm}(t) = C_{nm} e^{-c^2(n^2+m^2)t}}$

The <u>general solution</u>, by superposition, is

$$u(x,y,t) = \sum_{n,m\geq 0}^{\infty} f_{nm}(x,y)\, g_{nm}(t) \quad \text{(summing over all } \underline{\text{pairs}}\ n \ \&\ m\text{)}$$

i.e, $$\boxed{u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{nm} \sin nx \sin my\, e^{-c^2(n^2+m^2)t}}$$

Finally, use the initial condition (plug in $t=0$):

$$u(x,y,0) = \sum_{n,m\geq 1} b_{nm} \sin nx \sin my = \underline{\underline{2}} \sin x \sin 2y + \underline{\underline{3}} \sin 4x \sin 5y$$

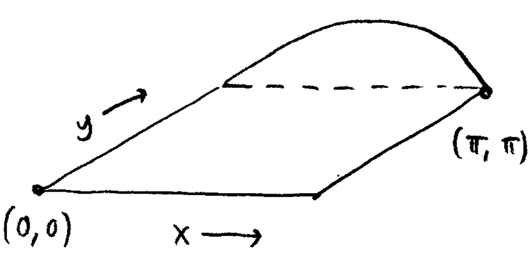
Equate coefficients: $b_{12} = 2$, $b_{45} = 3$, all other $b_{nm} = 0$.

Thus, the (unique) solution to the initial value problem is

$$\boxed{u(x,y,t) = 2 \sin x \sin 2y\, e^{-5c^2 t} + 3 \sin 4x \sin 5y\, e^{-41c^2 t}}$$

<u>Example 2(b)</u>: Heat equation with <u>non-zero</u> boundary conditions:

$$\begin{cases} u_t = c^2(u_{xx} + u_{yy}) \\ u(0,y,t) = u(\pi,y,t) = u(x,0,t) = 0 \\ u(x,y,0) = x(\pi - x) + \sin x \sin 2y + 3 \sin 4x \sin 5y \end{cases}$$



This is the IVP from Example 2(a) but with the boundary conditions from Example 1(a) (Laplace's equation) added.

* Thus, the unique solution to this initial/boundary value problem is just the sum of the solutions to Example 1(a) (the steady-state sol'n) and Example 2(a) (the homogeneous solution), i.e,

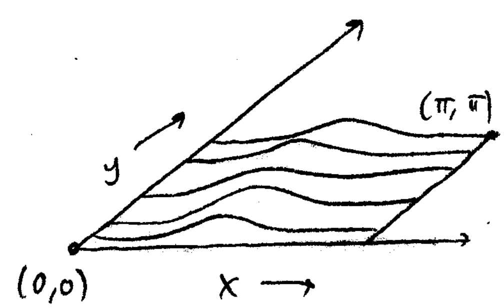
$$u(x,y,t) = u_0(x,y,t) + u_{ss}(x,y,t)$$

$$\boxed{u(x,y,t) = \underbrace{2\sin x \sin 2y\, e^{-5c^2t} + 3\sin 4x \sin 5y\, e^{-41c^2t}}_{\text{"Homogeneous" solution, i.e, the solution if all 4 boundary conditions are } 0^\circ \text{ (Sol'n to Example 2(a))}} + \underbrace{\sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3 \sinh n\pi} \sin nx \sinh ny}_{\text{Steady-state solution (sol'n to Example 1(a))}}}$$

Remark: In theory, any particular solution would do, but the steady-state is the only one we have any reasonable chance at finding.

Wave equation in 2D: $u_{tt} = c^2(u_{xx} + u_{yy})$.

Example 3: Let $u(x,y,t)$ = displacement of a point (x,y) on a square membrane of side-length π, and consider the following:



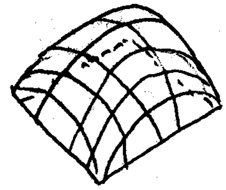
$u_{tt} = c^2(u_{xx} + u_{yy})$

$u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0$ (Boundary is immobile).

$u(x,y,t) = p(x)\, q(y)$ Initial wave (displacement)

$u_t(x,y,t) = 0$ Initial velocity (vertical).

Let's solve this if: $p(x) = x(\pi - x)$
$q(y) = y(\pi - y)$



Remark: Solving this is almost the same as solving the 2D heat equation.
The only difference in the general solution is $g_{nm}(t)$!

Observe: Assume $u(x,y,t) = f(x,y)\,g(t)$.
Plug back in: $f g'' = c^2 g f_{xx} + c^2 g f_{yy}$

$$\Rightarrow \frac{g''}{c^2 g} = \frac{f_{xx} + f_{yy}}{f} = \frac{\nabla^2 f}{f} = \lambda.$$

Get 2 equations: $\boxed{\begin{array}{l}\text{(i) } \nabla^2 f = \lambda f, \quad f(0,y) = f(\pi,y) = f(x,0) = f(x,\pi) = 0 \\ \text{(ii) } g'' = c^2 \lambda g\end{array}}$

Note that (i) is the same as before. Thus, $\boxed{\lambda = -(n^2 + m^2)}$

For (ii), we have $g'' = -c^2(n^2+m^2)\,g$

$$\Rightarrow g_{nm}(t) = A_{nm}\cos\left(c\sqrt{n^2+m^2}\,t\right) + B_{nm}\sin\left(c\sqrt{n^2+m^2}\,t\right).$$

But we also have an initial condition for $g(t)$:

$$u_t(x,y,0) = f(x,y)\,g'(0) = 0 \Rightarrow g'(0) = 0.$$

Thus, $g_{nm}(0) = B_{nm} = 0 \Rightarrow \boxed{g_{nm}(t) = A_{nm}\cos\left(c\sqrt{n^2+m^2}\,t\right)}$

We've shown that for any choice of n & m, we have a solution to the wave equation of the form:

$$u_{nm}(x,y,t) = f_{nm}(x,y)\,g_{nm}(t) = b_{nm}\sin nx \sin my \cos\left(c\sqrt{n^2+m^2}\,t\right)$$

Therefore, the general solution to the wave equation is:

$$\boxed{u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{nm} \sin nx \sin my \cos\left(c\sqrt{n^2+m^2}\,t\right)}$$

Note: Alternatively, we could write $\sum_{n,m \geq 1}$ instead of $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty}$.

Finally, use the other initial condition (plug in $t=0$):

$$u(x,y,0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{nm} \sin nx \sin my = \left(\sum_{n=1}^{\infty} B_n \sin nx\right)\left(\sum_{m=1}^{\infty} \beta_m \sin my\right).$$

$$= p(x)\,q(y) = x(\pi - x)\,y(\pi - y)$$

$$= \left(\sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3} \sin nx\right)\left(\sum_{m=1}^{\infty} \frac{4(1-(-1)^m)}{\pi m^3} \sin my\right)$$

Thus, $b_{nm} = B_n \beta_m = \dfrac{16(1-(-1)^n)(1-(-1)^m)}{\pi^2 n^3 m^3} = \begin{cases} \dfrac{64}{\pi^2 n^3 m^3} & n \,\&\, m \text{ even} \\ 0 & \text{otherwise} \end{cases}$

The particular solution to this initial/boundary value problem is:

$$\boxed{u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16(1-(-1)^n)(1-(-1)^m)}{\pi^2 n^3 m^3} \sin nx \sin my \cos\left(c\sqrt{n^2+m^2}\,t\right).}$$

Note: $\lim_{t \to \infty} u(x,y,t)$ doesn't exist. This makes sense because unlike heat, which diffuses, waves propagate forever.

Remark: Fix $(x,y) = (x_0, y_0)$. The result is a cosine wave - simple harmonic motion. So $u(x,y,t)$ is "a plane's worth of simple harmonic motion."