8. Nonlinear systems of differential equations

Motivation: "Most" ODE's & systems of ODE's that arise in practice are nonlinear. The most common technique of analysis is to "linearize" them first.

As before, most defining properties are determined by the eigenvalues & eigenvectors.

SIR model (Popular 3x3 system of ODE's)

This models an epidemic or disease in a population (e.g., flu)

Let $S(t)$ = # susceptible people at time t

$I(t)$ = # infected people at time t

$R(t)$ = # recovered people at time t

Initially, there are N suceptible (uninfected) people.

Transition: Susceptible $\longrightarrow$ Infected $\longrightarrow$ Recovered.

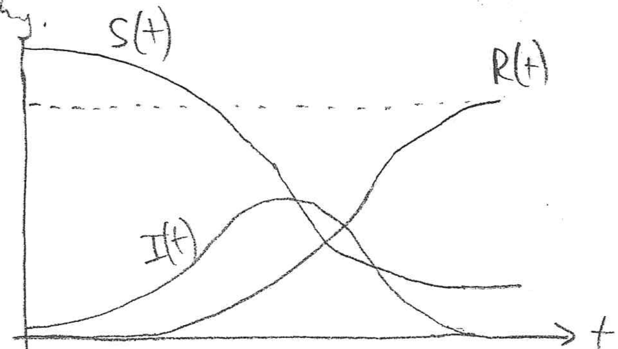
$\frac{dS}{dt} = -\alpha SI$ "proportional to both $S(t)$ & $I(t)$"

$\frac{dI}{dt} = \alpha SI - \gamma I$ "(rate people get sick) - (rate people get healthy)"

$\frac{dR}{dt} = \gamma I$ "rate people get healthy."

We get a nonlinear, autonomous system

$$\begin{cases} S' = -\alpha SI & S(0) = N \\ I' = \alpha SI - \gamma I & I(0) = 1 \\ R' = \gamma I & R(0) = 0 \end{cases}$$



component plots

Other epidemic models

- **SI model** (e.g., herpes, HIV)

$$S \xrightarrow{\alpha} I$$

$$\begin{cases} S' = -\alpha S I \\ I' = \alpha S I \end{cases}$$

- **SIS model**: Disease w/o immunity (e.g., chlamydia)

$$\begin{tikzcd} S \arrow[r, bend left, "\alpha"] & I \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

$$\begin{cases} S' = -\alpha S I - \gamma I \\ I' = \alpha S I - \gamma I \end{cases}$$

- **SIRS model**: Finite-time immunity (e.g., common cold)

$$\begin{tikzcd} S \arrow[r, "\alpha"] & I \arrow[r, "\gamma"] & R \arrow[ll, bend right, "\delta"'] \end{tikzcd}$$

$$\begin{cases} S' = -\alpha S I + \delta R \\ I' = \alpha S I - \gamma I \\ R' = \gamma I + \delta R \end{cases}$$

- **SEIR model**: E = exposed (incubation period, no symptoms)

$$S \xrightarrow{\alpha} E \xrightarrow{\varepsilon} I \xrightarrow{\gamma} R$$

$$\begin{cases} S' = -\alpha S I \\ E' = \alpha S I - \varepsilon E \\ I' = \varepsilon E - \gamma I \\ R' = \gamma I \end{cases}$$

- **SIR model** (with birth & death rate)

$$\begin{tikzcd} \arrow[r, "\beta"] & S \arrow[r, "\alpha"] \arrow[d, "\mu"] & I \arrow[r, "\varepsilon"] \arrow[d, "\mu"] & R \arrow[d, "\mu"] \\ & {} & {} & {} \end{tikzcd}$$

$$\begin{cases} S' = -\beta S I + \mu(N - S) \\ I' = \beta S I - \gamma I - \mu I \\ R' = \gamma I - \mu R \end{cases}$$

Competing species

Consider 2 species competing for a limited food supply & habitat.
(e.g., 2 types of fish in a pond).

$X(t)$ = population of Species 1

$Y(t)$ = population of Species 2.

Suppose that each species, without the other, would grow logistically.

$$\begin{cases} X' = r_1 X\left(1 - \frac{X}{M_1}\right) - s_1 XY & = X(\varepsilon_1 - \sigma_1 X - \alpha_1 Y) \\ Y' = r_2 Y\left(1 - \frac{Y}{M_2}\right) - s_2 XY & = Y(\varepsilon_2 - \sigma_2 Y - \alpha_2 Y). \end{cases}$$

$\alpha_1 Y$ = "decrease in growth rate of species 1 due to species 2"

So α_1 is the per capita decline.

Predator-Prey "Lotka-Volterra equations"

Consider 2 species, one of which depends on the other as a food source.

$X(t)$ = population of the prey

$Y(t)$ = population of the predator.

Assume that in the absense of the other species:

- the prey would grow exponentially: $X' = rX$
- the predator would decay exponentially. $Y' = -uY$.

Then we have the model: $\begin{cases} X' = rX - sXY \\ Y' = -uY + vXY \end{cases}$

This can be modified if the prey grows <u>logistically</u>:

$$\begin{cases} X' = rX(1 - X/M) - sXY \\ Y' = -uY + vXY. \end{cases}$$

This can also model a <u>host</u> $X(t)$ & a <u>parasite</u> $Y(t)$.

<u>Other population models</u>:

- Immune system vs. infective agent:

$X(t)$ = pop. of immune cells
$Y(t)$ = Level of infection

$$\begin{cases} X' = rY - sXY \\ Y' = uY - vXY \end{cases}$$

where: $-sXY$ = negative effect on immune system from fighting
$-vXY$ = limited effect on immune system in fighting
rY = immune response is proportional to infection level.

- Mutualism:

$X(t)$ = pop. of sharks
$Y(t)$ = pop. of feeder fish

$$\begin{cases} X' = rX(1 - X/M) + sXY \\ Y' = -uY + vXY \end{cases}$$

Other such relationships could include
- lichen & tree population
- leaf-cutter ants & fungi they cultivate
- fruit-trees & birds

Linearization & steady-state analysis

Example 1: $\begin{cases} X' = X(1-X-Y) \\ Y' = Y(0.75 - Y - 0.5X) \end{cases}$

First, find steady-state solutions (set $X' = Y' = 0$).

$$\begin{cases} 0 = X(1-X-Y) \\ 0 = Y(0.75-Y-0.5X) \end{cases} \qquad \begin{array}{l} X=0 \Rightarrow Y=0 \ \text{ or } \ \overbrace{0.75-Y-0.5X=0}^{Y=0.75} \\ Y=0 \Rightarrow X=0 \ \text{ or } \ \underbrace{1-X-Y=0}_{X=1} \end{array}$$

$$\text{or} \ \begin{cases} 1-X-Y=0 \\ 0.75-Y-0.5X=0 \end{cases} \Rightarrow X=Y=0.5$$

We have 4 steady-state solutions, (X^*, Y^*):

- $(0,0)$: both species are extinct
- $(1,0)$: Species 2 is extinct $[K_1 = 1]$
- $(0, .75)$: Species 1 is extinct $[K_2 = .75]$
- $(.5, .5)$: Both species co-exist.

Graphical Interpretation: Recall that an isocline is a curve or line where $X' = 0$ or $Y' = 0$. (Also called "nullcline").

$\boxed{X' = 0}$: $X = 0$ or $1 - X - Y = 0$

$\boxed{Y' = 0}$: $Y = 0$ or $0.75 - Y - 0.5X = 0$

The steady-state solutions are the intersections of X-isoclines with Y-isoclines.

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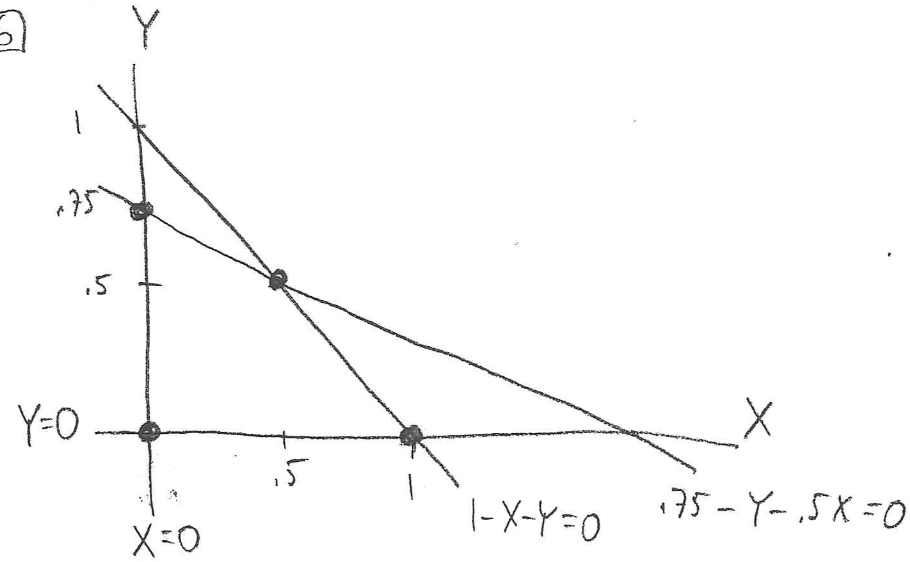


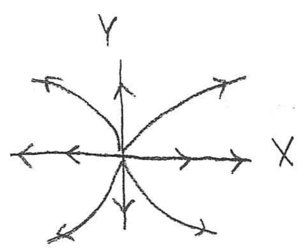
Next, we'll analyze the behavior of the phase portrait near each steady-state point using a technique called <u>linearization</u>.

$\boxed{(X^*, Y^*) = (0,0)}$ $\quad \begin{cases} X' = X - X^2 - XY \approx X \\ Y' = .75Y - Y^2 - .5XY \approx .75Y \end{cases}$ if $X, Y \approx 0$

The system linearized at $(0,0)$ is $\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & .75 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$

$\lambda_1 = 1 \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = .75 \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



To linearize at a non-zero critical point (X^*, Y^*), we must <u>change variables</u>:

let $\begin{cases} P = X - X^* \\ Q = Y - Y^* \end{cases} \Rightarrow \begin{cases} X = P + X^* \\ Y = Q + Y^* \end{cases} \quad \begin{matrix} P' = X' \\ Q' = Y' \end{matrix}$

<u>Plug back into</u>: $\boxed{\begin{matrix} X' = X(1 - X - Y) \\ Y' = Y(.75 - Y - .5X) \end{matrix}}$

$$\begin{cases} P' = (P+X^*)(1-P-Q-X^*-Y^*) \\ Q' = (Q+Y^*)(.75-Q-Y^*-.5P-.5X^*) \end{cases}$$

$P =$ dist. from X^*

$Q =$ dist. from Y^*

Simplifying...

$$P' = (1-2X^*-Y^*)\,P - X^*Q + (\text{non-linear terms})$$

$$Q' = .5U + (.75-.5X^*-2Y^*)V + (\text{non-linear terms})$$

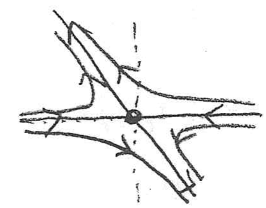
assume ≈ 0, since $P, Q \approx 0$.

$$\boxed{\begin{pmatrix} P' \\ Q' \end{pmatrix} = \begin{pmatrix} 1-2X^*-Y^* & -X^* \\ -.5Y^* & .75-2Y^*-.5X^* \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix} + \begin{pmatrix} \text{non-linear} \\ \text{terms} \end{pmatrix}}$$

Back to linearizing at the other 3 steady-states:

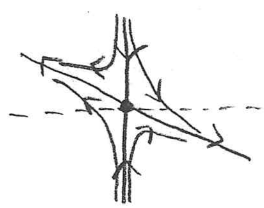
$\boxed{(X^*, Y^*) = (1,0)}$

$$\begin{pmatrix} P' \\ Q' \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & .25 \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix} \qquad \lambda_1 = -1 \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \lambda_2 = .25 \quad v_2 = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$$



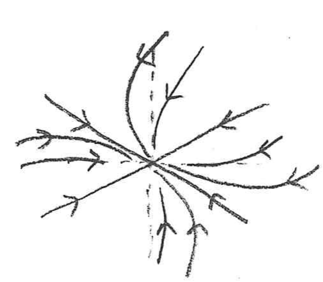
$\boxed{(X^*, Y^*) = (0, .75)}$

$$\begin{pmatrix} P' \\ Q' \end{pmatrix} = \begin{pmatrix} .25 & 0 \\ .375 & -.75 \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix} \qquad \lambda_1 = .25 \quad v_1 = \begin{pmatrix} 8 \\ -3 \end{pmatrix} \qquad \lambda_2 = -.75 \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

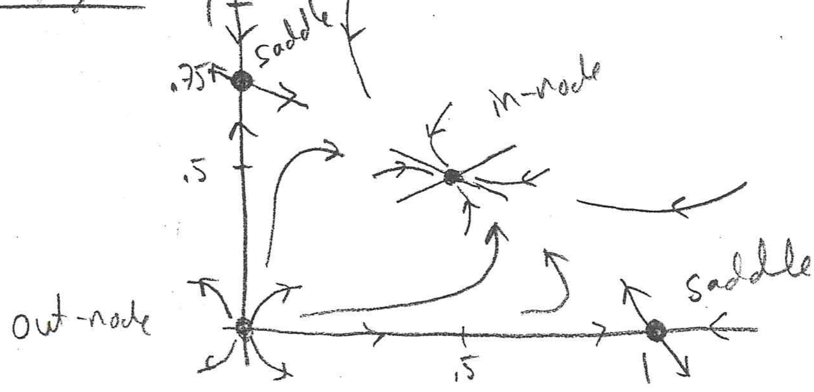


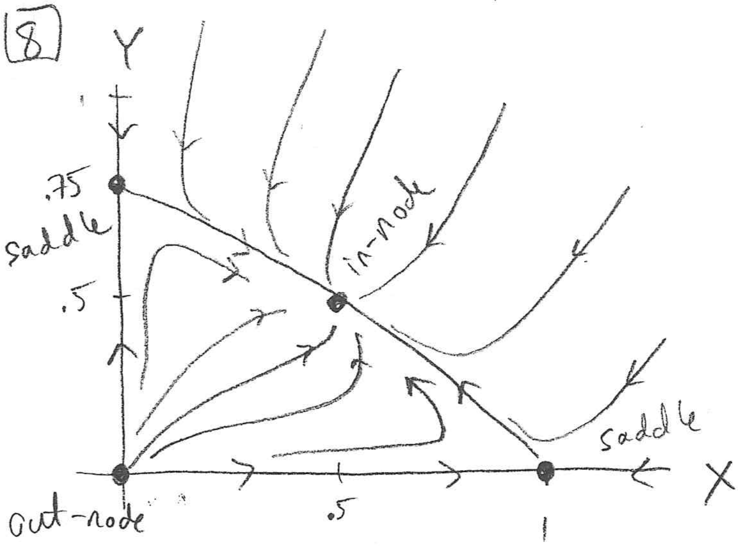
$\boxed{(X^*, Y^*) = (.5, .5)}$

$$\begin{pmatrix} P' \\ Q' \end{pmatrix} = \begin{pmatrix} -.5 & -.5 \\ -.25 & -.5 \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix} \qquad \lambda_1 \approx -.146 \quad v_1 = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} \qquad \lambda_2 \approx -.854 \quad v_2 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$



<u>Putting this together</u>



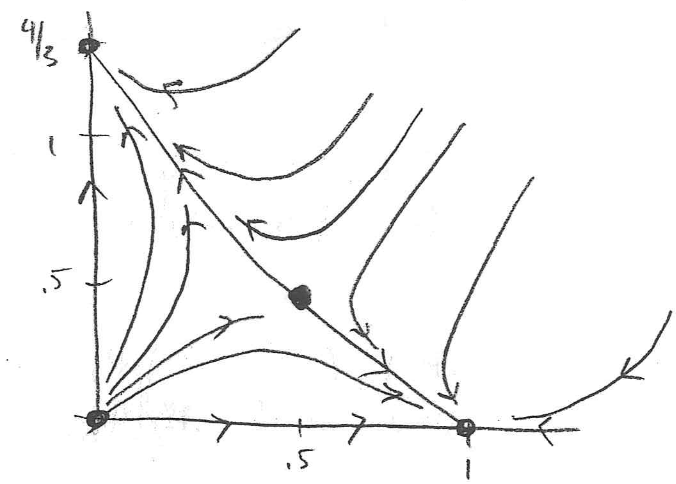


$(X^*, Y^*) = (.5, .5)$ is a <u>stable</u> steady-state

<u>Example 2</u>: Consider the following: $\begin{cases} X' = X(1 - X - Y) \\ Y' = Y(.8 - .6Y - X) \end{cases}$

It is easy to check that there are 4 steady-states:

- $(0,0)$ out-node
- $(1,0)$ in-node
- $(0, 4/3)$ in-node
- $(\frac{1}{2}, \frac{1}{2})$ saddle



(X^*, Y^*) is an <u>unstable</u> steady-state.

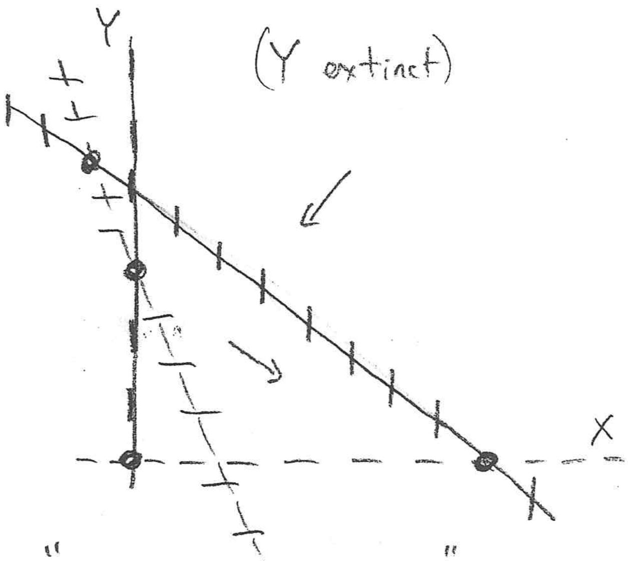
<u>Question</u>: What's going on? Are there other possibilities?

<u>Answer</u>: Consider the system $\begin{cases} X' = X(\varepsilon_1 - \sigma_1 X - \alpha_1 Y) \\ Y' = Y(\varepsilon_2 - \sigma_2 Y - \alpha_2 X) \end{cases}$

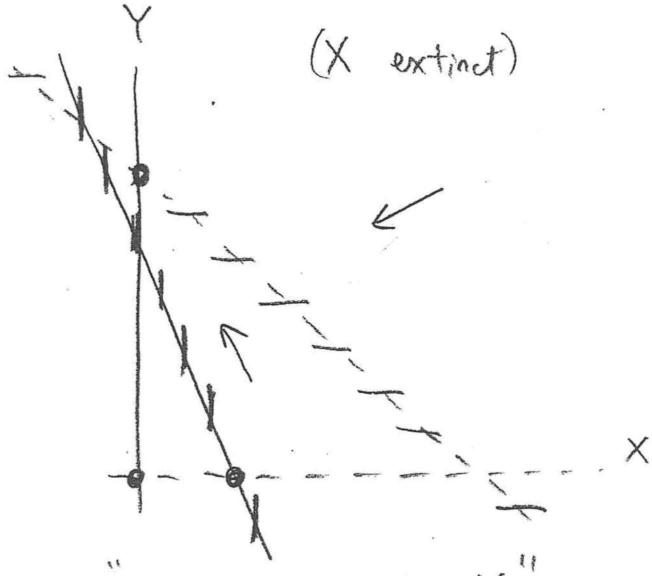
X-isoclines: $X = 0$, $\varepsilon_1 - \sigma_1 X - \alpha_1 Y = 0$ "solid"

Y-isoclines: $Y = 0$, $\varepsilon_2 - \sigma_2 Y - \alpha_2 X = 0$ "dashed"

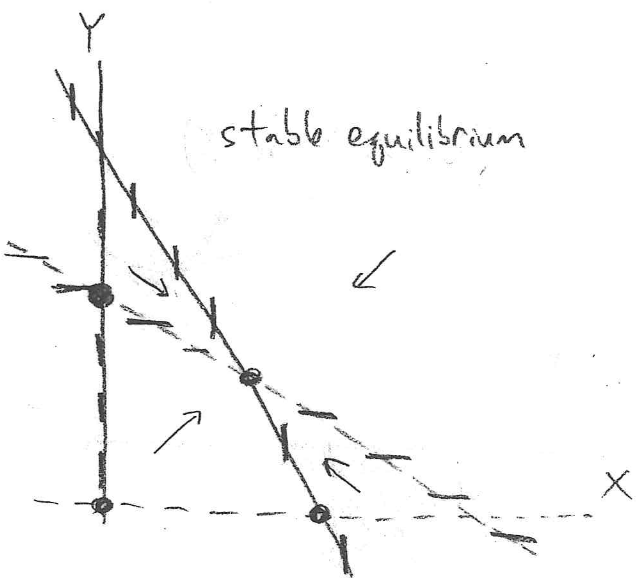
There are 4 types of dynamics



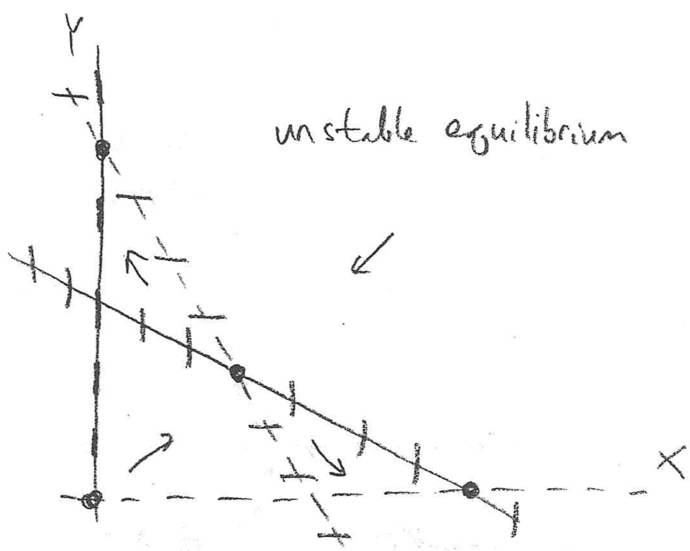
"X outcompetes Y"



"Y out competes X"

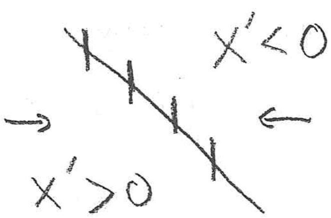


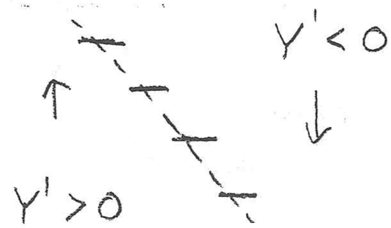
"coexistence"



"one species outcompetes the other. Depends on init. cond."

Note:

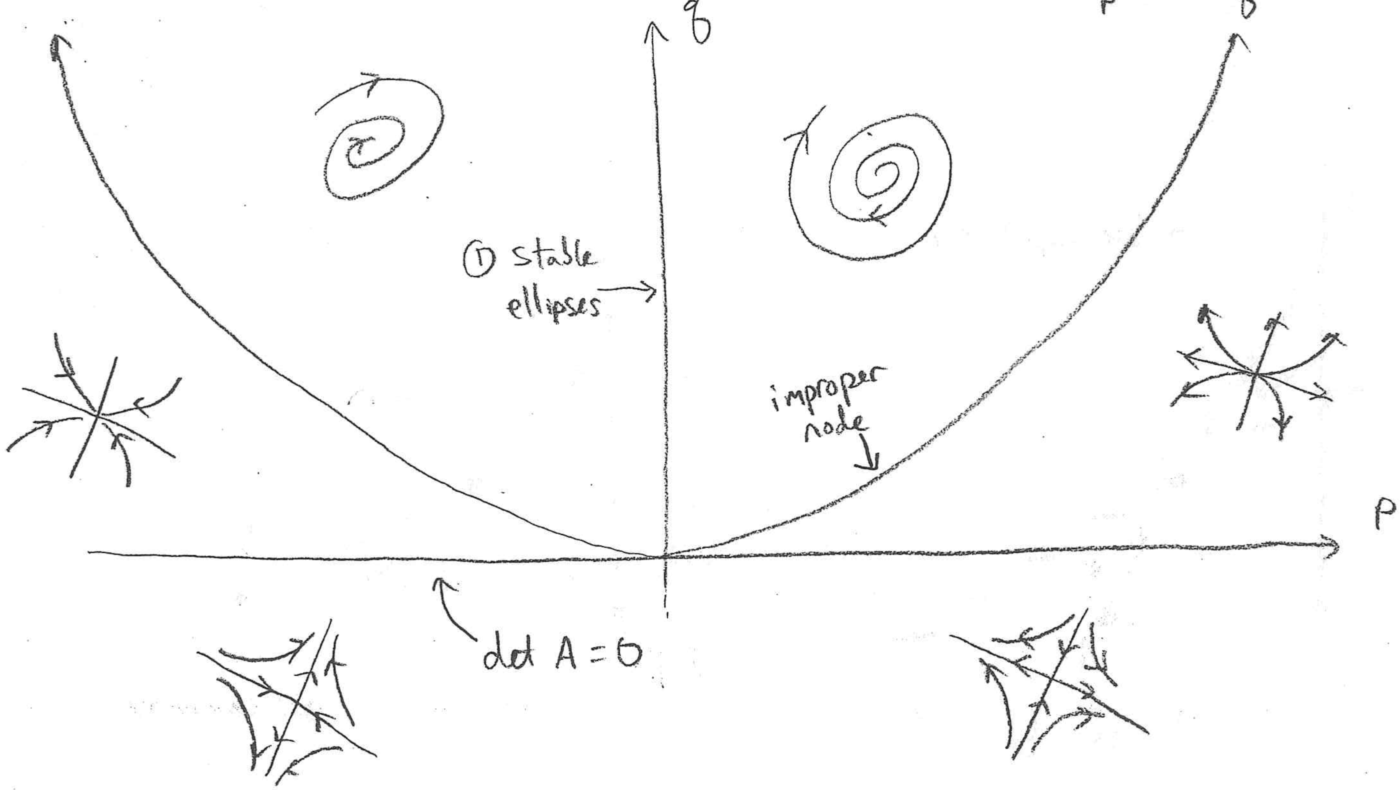




10 When might linearization fail?

We linearize a system at a steady-state point expecting that the dynamics near it is approximated by the linearized system.

There are a few "special cases" when this fails, and they all correspond to "boundaries" between regions in the following picture (recall Section 4): $\lambda^2 - \underbrace{\mathrm{tr}\,A}_{p} + \underbrace{\det A}_{q} = 0$



① stable ellipses 

② improper nodes 

③ det A = 0 

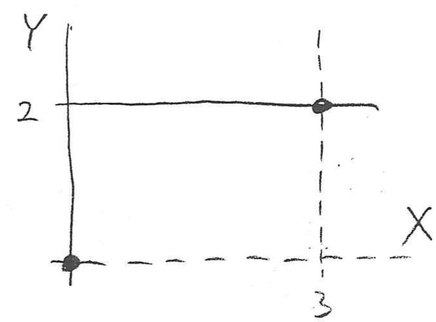
in all of these cases, the non-linear terms could cause the system to "fall" on either side of the boundary.

Predator-prey models

Example 3: Consider the following: $\begin{cases} X' = X - \frac{1}{2}XY = X(1-\frac{1}{2}Y) \\ Y' = -\frac{3}{4}Y + \frac{1}{4}XY = Y(-\frac{3}{4}+\frac{1}{4}X) \end{cases}$

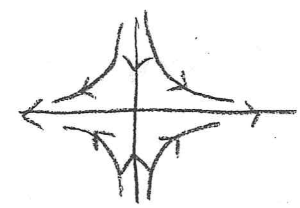
Steady-state solutions: $\begin{cases} 0 = X(1-\frac{1}{2}Y) \\ 0 = Y(-\frac{3}{4}+\frac{1}{4}X) \end{cases}$ $\quad (X^*, Y^*) = (0,0)$ or $(3,2)$

- X-isoclines: $X=0$, $Y=2$
- Y-isoclines: $Y=0$, $X=3$



Linearize at $(X^*, Y^*) = (0,0)$:

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -3/4 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} \qquad \begin{matrix} \lambda_1 = 1 & v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \lambda_2 = -\frac{3}{4} & v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{matrix}$$



Linearize at $(X^*, Y^*) = (3,2)$:

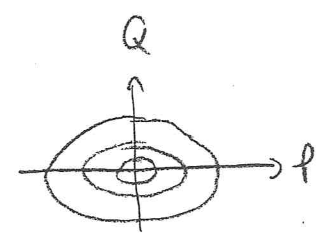
Let $\begin{cases} P = X-3 \\ Q = Y-2 \end{cases} \Rightarrow \begin{matrix} X = P+3 \\ Y = Q+2 \end{matrix} \Rightarrow \begin{matrix} P' = (P+3)(1-\frac{1}{2}Q - 1) = -\frac{3}{2}Q + (\text{nonlinear}) \\ Q' = (Q+2)(-\frac{3}{4}+\frac{1}{4}P+\frac{3}{4}) = \frac{1}{2}P + (\text{nonlinear}) \end{matrix}$

$$\begin{pmatrix} P' \\ Q' \end{pmatrix} = \begin{pmatrix} 0 & -3/2 \\ 1/2 & 0 \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix}$$

$\lambda_{1,2} = \pm\frac{\sqrt{3}}{2}i$; this is the "problem case", we can't determine if the phase space will be inward or outward spirals, or stable.

$$\frac{dQ}{dP} = \frac{dQ/dt}{dP/dt} = \frac{\frac{1}{2}P}{-\frac{3}{2}Q} = \frac{-P}{3Q}$$

$$\Rightarrow P\,dP + 3Q\,dQ = 0 \xrightarrow[\text{integrate}]{} P^2 + 3Q^2 = k$$



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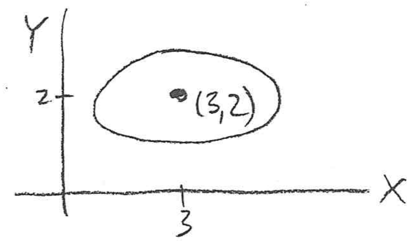
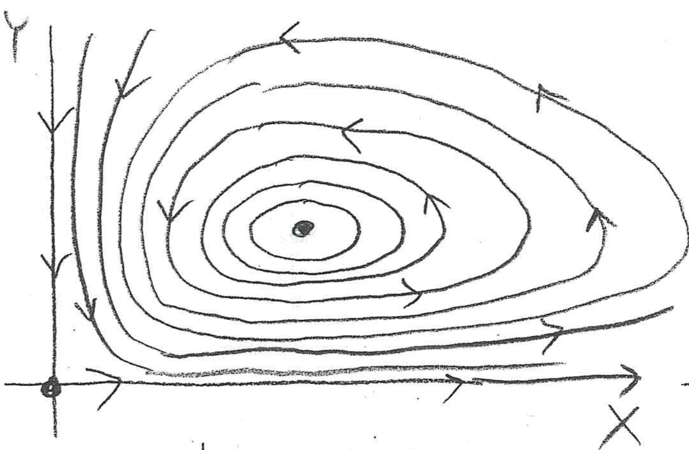
Return to the nonlinear system:

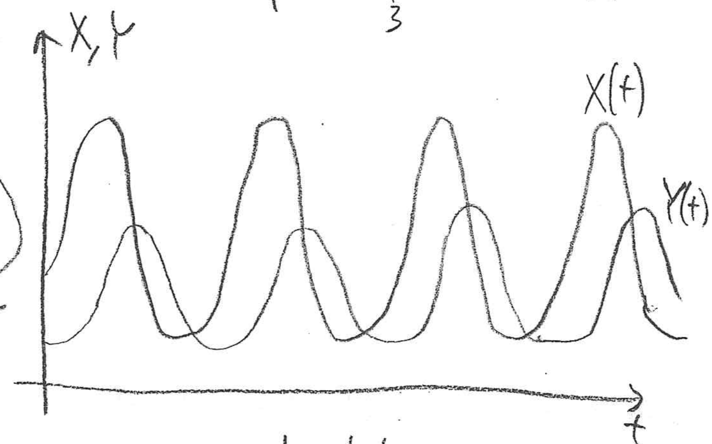
$$\frac{dY}{dX} = \frac{Y'}{X'} = \frac{Y\left(-\frac{3}{4}+\frac{1}{4}X\right)}{X\left(1-\frac{1}{2}Y\right)} \implies \frac{1-\frac{1}{2}Y}{Y}\,dY = \frac{-\frac{3}{4}+\frac{1}{4}X}{X}\,dX$$

$$\xrightarrow[\text{Integrate}]{} \quad \frac{3}{4}\ln X + \ln Y - \frac{1}{2}Y - \frac{1}{4}X = C$$





phase portrait



component plots

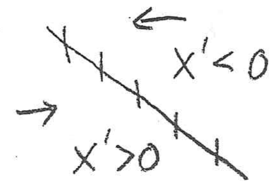
Example 4: Prey grows logistically.

$$\begin{cases} X' = rX(1-X/M) - sXY \\ Y = -uY - vXY \end{cases}$$

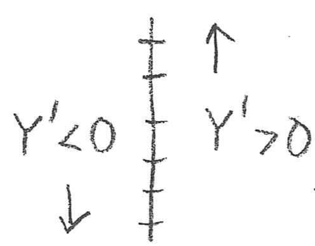
(let's "normalize" & set $M=1$)

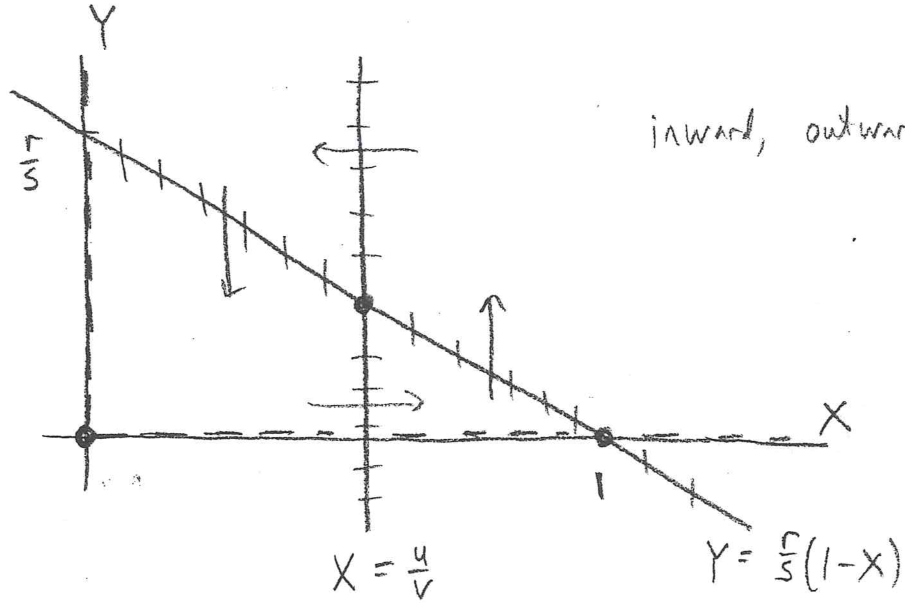
Steady-state: $\begin{cases} 0 = X\left(r(1-X) - sY\right) \\ 0 = Y(-u+vX) \end{cases}$ $\quad (X^*, Y^*) = (0,0),\ (1,0),\ \left(\frac{u}{v}, \frac{r}{s}\left(1-\frac{u}{v}\right)\right)$

- X-isoclines: $X = 0$ and $r(1-X) - sY = 0$



- Y-isoclines: $Y=0$ and $-u+vX=0$





inward, outward, or stable???

Application: Suppose $X(t)$ = crop
$Y(t)$ = pest or weed.

A Farmer might try to grow a "superior cultivar" with a higher growth rate r, to attempt to "outgrow" the pest.

But this will only increase Y^*, & leave X^* unchanged!

Linerization (to determine stability of the equilibrium point)

To make things easier, let's set $r=1.3$, $s=.5$, $u=.7$, $v=1.6$

$$\begin{cases} X' = 1.3X(1-X) - .5XY \\ Y' = -.7Y + 1.6XY \end{cases} \Rightarrow (X^*, Y^*) = (0,0),\ (1,0),\ (.4375,\ 1.4625).$$

Change vars: $\begin{cases} P = X - .4375 \\ Q = Y - 1.4625 \end{cases} \Rightarrow \begin{array}{l} X = P + .4375 \\ Y = Q + 1.4625 \end{array}$

Plug in: $\begin{cases} P' = .56875P - .21875Q - 1.3P^2 - .5PQ \\ Q' = 2.34P + 1.6PQ \end{cases}$

$$\begin{pmatrix} P' \\ Q' \end{pmatrix} = \begin{pmatrix} .56875 & -.21875 \\ 2.34 & 0 \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} \qquad \lambda_{1,2} = -.2844 \pm .6565i$$

inward spirals!

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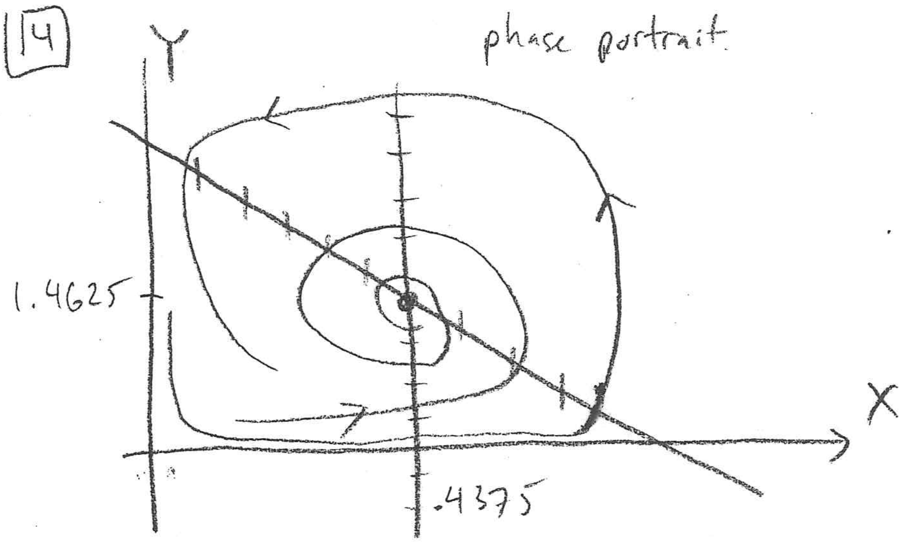

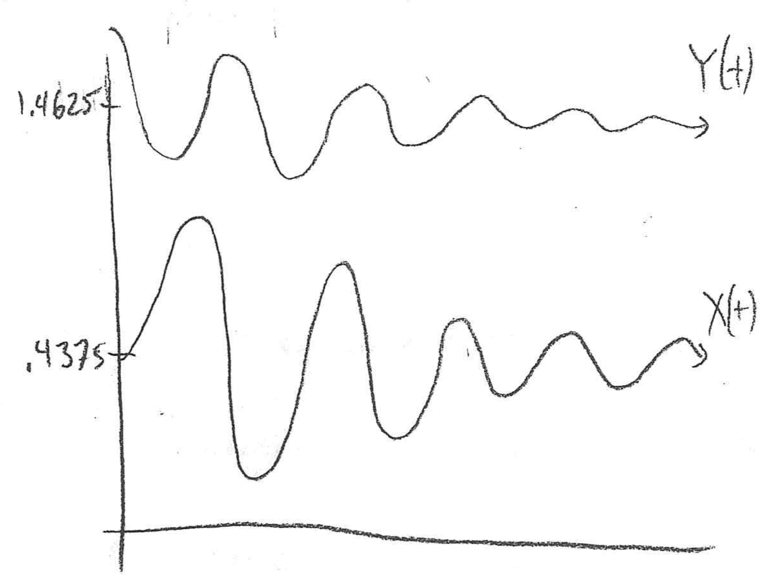

component plots